

# SINGULARITIES OF SPACELIKE CONSTANT MEAN CURVATURE SURFACES IN LORENTZ-MINKOWSKI SPACE

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**ABSTRACT.** We study singularities of spacelike, constant (non-zero) mean curvature (CMC) surfaces in the Lorentz-Minkowski 3-space  $L^3$ . We show how to solve the singular Björling problem for such surfaces, which is stated as follows: given a real analytic null-curve  $f_0(x)$ , and a real analytic null vector field  $v(x)$  parallel to the tangent field of  $f_0$ , find a conformally parameterized (generalized) CMC  $H$  surface in  $L^3$  which contains this curve as a singular set and such that the partial derivatives  $f_x$  and  $f_y$  are given by  $\frac{df_0}{dx}$  and  $v$  along the curve. Within the class of generalized surfaces considered, the solution is unique and we give a formula for the generalized Weierstrass data for this surface. This gives a framework for studying the singularities of non-maximal CMC surfaces in  $L^3$ . We use this to find the Björling data – and holomorphic potentials – which characterize cuspidal edge, swallowtail and cuspidal cross cap singularities.

## 1. INTRODUCTION

Spacelike constant mean curvature (CMC) surfaces in  $(2+1)$ -dimensional space-time  $\mathbb{L}^3$  were studied in [5] and [12] using a generalized Weierstrass representation whereby the surface is represented by a holomorphic map into a loop group. This is an application of the method of Dorfmeister, Pedit and Wu (DPW) [7] for harmonic maps into symmetric spaces. In the non-compact case, the Iwasawa decomposition of the loop group, used to construct the solutions, is only valid on an open dense set, the *big cell*. It was shown in [5] that singularities of the CMC surface arise as the boundary of the big cell is encountered. Here we will analyze these singularities and show how to construct CMC surfaces with prescribed singular curves, and prescribed types of singularities, via a singular Björling formulation.

One of the obstructions to the effective use of integrable systems methods for solving global problems in geometry has been the break-down of the loop group decompositions used to construct solutions. A motivating factor here is to understand and make use of the big cell boundary behaviour.

**1.1. Singularities of maximal surfaces and fronts.** In the context of surfaces in Euclidean 3-space  $\mathbb{E}^3$ , a *frontal* is a differentiable map  $f : M^2 \rightarrow \mathbb{E}^3$ , from a surface  $M$ , which has a well defined normal direction, that is, a map  $\mathbf{n}_E : M^2 \rightarrow \mathbb{S}^2 \subset \mathbb{E}^3$  which is orthogonal to  $f_*(TM^2)$ . If the map  $(f, \mathbf{n}_E)$  is an immersion, then  $f$  is called a (*wave*) *front*. A singular point of any smooth map  $f : M^2 \rightarrow \mathbb{E}^3$  is one where  $f$  is not immersed, and singular points  $p_1$  and  $p_2$  of  $f_1 : M_1^2 \rightarrow \mathbb{E}^3$  and  $f_2 : M_2^2 \rightarrow \mathbb{E}^3$  are called diffeomorphically equivalent if there exist local diffeomorphisms of the corresponding spaces which commute with these maps. A theory of the singularities of fronts can be found in Arnold [3]. Geometric concepts, such as curvature and completeness, for surfaces with singularities have been defined by Saji, Umehara and Yamada in [17].

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In this article we will encounter three standard singularities: the *cuspidal edge*, given by  $f(u, v) = (u^2, u^3, v)$ , the *swallowtail* given by  $(3u^4 + u^2v, 4u^3 + 2uv, v)$  and the *cuspidal cross cap* given by  $(u, v^2, uv^3)$  (Figure 1). The first two singularities are fronts, but the third is only a frontal.



FIGURE 1. Left to right: Cuspidal edge, swallowtail and cuspidal cross cap.

A point to note is that if one wants a sensible theory of singularities, for example if one would like to classify singularities for a specific type of surface, then one needs to consider *generic* singularities, that is singularities which persist under continuous deformations of the surface through the appropriate class. If one considers the class of  $\mathcal{C}^\infty$  maps of 2-manifolds into 3-manifolds, Whitney showed that generic singularities are cross caps [20].

Fronts and frontals arise naturally within the context of integrable systems – very often it is exactly such surfaces, rather than immersions, which are produced via loop group constructions. Conversely, for many geometric problems, it is more or less unavoidable to consider surfaces with singularities: for example it is well known that there is no complete immersion of the hyperbolic plane into  $\mathbb{E}^3$ , and for the case of spacelike maximal (mean curvature zero) surfaces in  $\mathbb{L}^3$  the only complete immersion is the plane. For these two examples, generic singularities have been classified: for constant Gauss curvature surfaces in  $\mathbb{E}^3$ , Ishikawa and Machida [13] showed that they consist of cuspidal edges and swallowtails; for maximal surfaces in  $\mathbb{L}^3$ , Fujimori, Saji, Umehara and Yamada [19, 11] showed that the generic singularities are all three of those shown in Figure 1.

Recently there have been a number of interesting studies of maximal surfaces and their singularities: the reader is referred to articles such as [2, 9, 8, 10, 15, 11] and the references therein. Most closely related to the present article are the classification of generic singularities [19, 11] already mentioned, and the work of Y.W. Kim and S.D. Yang [15] on the singular Björling problem for maximal surfaces.

**1.2. The Björling problem.** The classical Björling problem for minimal surfaces in  $\mathbb{E}^3$  is to find the unique minimal surface containing a given real analytic curve with prescribed tangent planes along the curve (see [6]). The solution is obtained from the initial data by an analytic extension and an elementary formula in terms of integrals. Since the solution is tied to the Weierstrass representation of minimal surfaces in terms of holomorphic data, one has a similar construction for regular maximal surfaces in  $\mathbb{L}^3$ , given in [2], which also have such a holomorphic representation. More generally, Kim and Yang [15] show that there is also a solution when the initial curve is null (which implies that the surface is not immersed there). Instead of prescribing the tangent plane along the curve, one seeks a surface which is conformally immersed except along the curve, with coordinates  $z = x + iy$ , and where the curve is given by  $\{y = 0\}$ , and then prescribes the value of  $f_y$ , a null vector field parallel to  $f_x$ . Note that null vectors are orthogonal if and only if they are parallel, so this makes sense in terms of the conformal coordinates. One can then use this construction to study the singularities of maximal surfaces.

As a generalization of the Weierstrass representations for minimal and maximal surfaces, one has the DPW method for CMC  $H \neq 0$  surfaces in both  $\mathbb{E}^3$  and  $\mathbb{L}^3$ . In [4], it was shown that one could use this method to solve the generalization of the Björling problem to non-minimal CMC surfaces in  $\mathbb{E}^3$ . It is clear that essentially the same construction works for *regular* CMC  $H \neq 0$  surfaces in  $\mathbb{L}^3$ , and we will show below that the *singular* Björling problem can also be solved for non-maximal CMC surfaces. The main obstacle which needs to be circumvented is that the DPW method depends on the use of an  $SU_{1,1}$  frame (extended to the loop group) and then a loop group decomposition to go to the holomorphic data. This  $SU_{1,1}$  frame is not defined along the singular curve, because the (Lorentzian) unit normal becomes lightlike and blows up. Below, we will get around this by defining a special  $SU_{1,1}$  “frame”, called the *singular frame*, along the curve, the definition of which is motivated by our analysis of the loop group construction.

**1.3. The DPW method.** The generalized Weierstrass representation for spacelike CMC surface in  $\mathbb{L}^3$  follows the same logic as that for CMC surface in Euclidean 3-space: in the maximal case, where the mean curvature  $H$  is zero, there is a Weierstrass representation in terms of a pair of holomorphic functions, just as for minimal surfaces, related to the fact that the Gauss map is holomorphic. For the non-maximal case, the Gauss map is harmonic but not holomorphic, and one can instead use the holomorphic representation for harmonic maps given in [7]. The only real difference from the Euclidean case is the non-compactness of the isometry group, leading to an incomplete picture of what is actually constructed from the given holomorphic data. For more details and references, see [5].

The DPW construction described in [5] is as follows: A CMC  $H$  immersion  $f : \Sigma \rightarrow \mathbb{L}^3$  from a Riemann surface into Minkowski 3-space can be represented by a certain type of holomorphic map  $\hat{\Phi} : \Sigma \rightarrow \Lambda SL(2, \mathbb{C})_\sigma$  into the twisted loop group of smooth maps from the unit circle into  $SL(2, \mathbb{C})$ . The map  $\hat{\Phi}$  is called a *holomorphic extended frame* for  $f$ . In connection with the Iwasawa decomposition with respect to the non-compact real form  $\Lambda SU_{1,1}$ , the loop group  $\Lambda SL(2, \mathbb{C})_\sigma$  can be written as a disjoint union  $\mathcal{B}_{1,1} \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \dots$ . The set  $\mathcal{B}_{1,1}$  is open and dense in  $\Lambda SL(2, \mathbb{C})_\sigma$ , and is called the (Iwasawa) big cell. As a converse to the above statement concerning  $f$ , given a holomorphic extended frame, if we restrict to  $\Sigma^\circ := \hat{\Phi}^{-1}(\mathcal{B}_{1,1})$ , one obtains a CMC  $H$  immersion into  $\mathbb{L}^3$ . Behaviour of the surface as the largest two small cells,  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , are approached was examined in [5], and it was shown that the CMC surface extends continuously to  $\hat{\Phi}^{-1}(\mathcal{P}_1)$ , but is not immersed there, and that the surface blows up as  $\hat{\Phi}^{-1}(\mathcal{P}_2)$  is approached.

**1.4. Results of this article.** As we are interested in finite singularities, we define a *generalized CMC  $H$  surface* to be a map  $f$  obtained from a holomorphic extended frame  $\hat{\Phi}$ , restricted to  $\Sigma_s := \hat{\Phi}^{-1}(\mathcal{B}_{1,1} \cup \mathcal{P}_1)$ . This includes all regular CMC  $H$  surfaces, as one can always find a holomorphic extended frame for a regular surface which takes values in the big cell  $\mathcal{B}_{1,1}$ . We know that the singular set  $C := \hat{\Phi}^{-1}(\mathcal{P}_1)$ , where  $f$  is not immersed, is locally given as the zero set of a non-constant real analytic function. We say that  $z_0 \in C$  is *weakly non-degenerate* if  $\hat{\Phi}$  maps some open curve containing  $z_0$  into  $\mathcal{P}_1$ . This is simply the weakest condition needed to consider the singular Björling construction, and holds for a generic point in  $C$ .

The main results of this article can be summarized as Theorem 4.1, Theorem 5.7 and Theorem 5.9. The first of these results is the solution of the singular Björling problem for CMC surfaces in  $\mathbb{L}^3$ . It essentially says that given a real analytic curve  $f_0 : J \rightarrow \mathbb{L}^3$ , from some interval  $J \subset \mathbb{R} \subset \mathbb{C}$ , such that  $\frac{df_0}{dx}$  is a null vector field, and given a real analytic vector field  $v : J \rightarrow \mathbb{L}^3$  which is proportional to  $\frac{df_0}{dx}$ , then, for any constant  $H > 0$ , there is a unique, weakly non-degenerate, generalized CMC  $H$  surface  $f$  satisfying  $f|_J = f_0$  and  $\frac{\partial f}{\partial y}|_J = v$ . It also gives

a formula for the holomorphic potential for the surface in terms of analytic extensions of the data specified along  $J$ .

The other two results mentioned, Theorems 5.7 and 5.9, give the conditions on the Björling data for the singularity at a point  $z_0 \in J$  to be diffeomorphic to a cuspidal edge, swallowtail or cuspidal cross cap. The conditions are simple: for the given Björling data, one can always write  $\frac{df_0}{dx} = s[\cos \theta, \sin \theta, 1]$  and  $v(x) = t[\cos \theta, \sin \theta, 1]$ , where  $s$ ,  $t$ , and  $\theta$  are  $\mathbb{R}$ -valued, and we assume that  $s$  and  $t$  do not vanish simultaneously to avoid branch points. Then  $s(0) \neq 0 \neq t(0)$  corresponds to a cuspidal edge at the coordinate origin;  $s(0) = 0$  and  $s'(0) \neq 0$  corresponds to a swallowtail;  $t(0) = 0$  and  $t'(0) \neq 0$  is a cuspidal cross cap (see Figure 2).

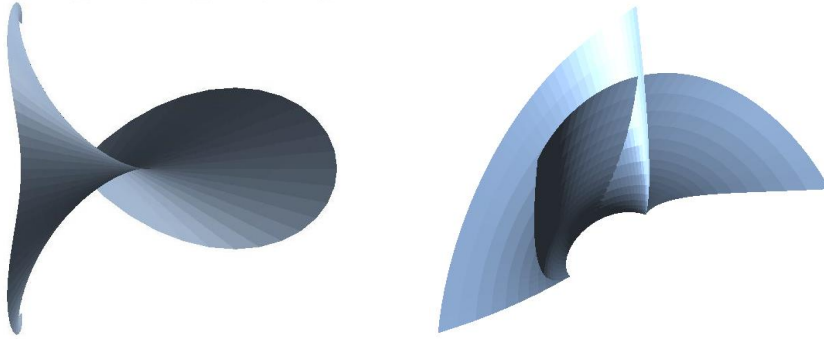


FIGURE 2. Left: a CMC swallowtail singularity, computed numerically from the Björling data  $s(x) = x$ ,  $t(x) = 1$ ,  $\theta(x) = 0.0001x$ . Right: a CMC cuspidal cross cap, computed from the data  $s(x) = 1 - x$ ,  $t(x) = x$ ,  $\theta(x) = 0.001x$ . The images have been rescaled in the direction  $e_2 + e_3$ .

**1.5. Open questions.** It appears plausible that the three types of singularities just mentioned are the generic singularities for CMC surfaces in  $\mathbb{L}^3$ , just as was shown for maximal surfaces in [11]. To prove this using the constructions here, one would first need to show that generic singularities do not occur on higher small cells  $\mathcal{P}_j$ , for  $j > 2$ . This seems likely, because the codimensions of the small cells  $\mathcal{P}_j$  in the loop group increase (pairwise) as  $j$  increases. Regardless of genericity, knowledge of the behaviour of the surface close to such points would also be interesting to have.

**1.6. Alternative approaches: the Kenmotsu formula representation.** An alternative to the DPW method is the Kenmotsu formula [14] for CMC surfaces in  $\mathbb{E}^3$ , adapted to spacelike CMC surfaces in  $\mathbb{L}^3$  by Akutagawa and Nishikawa in [1]. This is also a generalization of the Weierstrass representation for minimal/maximal surfaces, as a formula in terms of the harmonic Gauss map. In contrast to the DPW method, one is still left with the problem of constructing the harmonic map. The Kenmotsu-Akutagawa-Nishikawa approach has been used by Y. Umeda [18] to study CMC surfaces with singularities in  $\mathbb{L}^3$ , giving the conditions on the harmonic Gauss map corresponding to cuspidal edges, swallowtails and cuspidal cross caps, as well as some examples. It is stated as an open problem whether or not a CMC cuspidal cross cap exists: here we give a positive answer to this question, and, in principal, construct all such singularities from their Björling data.

## 2. BACKGROUND MATERIAL

This section is a short summary of results in [5]. We use mostly the same notation and definitions here. **Notational convention:** If  $\hat{X}$  is some object with values in the loop group, with loop parameter  $\lambda$ , then dropping the hat means the object is evaluated at  $\lambda = 1$ :

$$X := \hat{X} \Big|_{\lambda=1}.$$

**2.1. The loop group formulation for CMC surfaces in  $\mathbb{L}^3$ .** We use the basis

$$e_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

for the Lie algebra  $\mathfrak{su}_{1,1}$ . With respect to the Killing metric,  $\langle X, Y \rangle = \frac{1}{2} \text{trace}(XY)$ , these vectors are orthogonal and normalized as follows:

$$\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle = 1,$$

so we identify  $\mathfrak{su}_{1,1}$  with the Lorentz-Minkowski space  $\mathbb{L}^3 = \mathbb{R}^{2,1}$ , and also use the notation  $[a, b, c]^T = ae_1 + be_2 + ce_3$  for a point in  $\mathbb{L}^3$ .

Let  $G$  be the subgroup of  $SL(2, \mathbb{C})$  consisting of elements of either  $SU_{1,1}$  or of  $ie_1 \cdot SU_{1,1}$ ,

$$(2.1) \quad G = \left\{ \begin{pmatrix} a & b \\ \varepsilon \bar{b} & \varepsilon \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, \varepsilon(a\bar{a} - b\bar{b}) = 1, \varepsilon = \pm 1 \right\}.$$

The Lie algebra of  $G$  is  $\mathfrak{g} = \mathfrak{su}_{1,1}$ .

The twisted loop group  $\mathcal{U} := \Lambda G_\sigma$  consists of maps,  $x : \mathbb{S}^1 \rightarrow G$ , from the unit circle into  $G$ , such the diagonal and off-diagonal elements of the matrix are even and odd functions of the  $\mathbb{S}^1$  parameter  $\lambda$ . All loops are of a suitable smoothness class so that the loop groups are Banach Lie groups. An element of  $\mathcal{U}$  can again be written as in (2.1), where now  $a$  and  $b$  are respectively even and odd functions of  $\lambda$ . We will generally be considering loops which extend holomorphically to an annulus around  $\mathbb{S}^1$ , and for these the holomorphic extensions of  $\bar{a}$  and  $\bar{b}$  respectively have Fourier expansions  $a^*(\lambda) := (a(1/\bar{\lambda}))$  and  $b^*(\lambda) := (b(1/\bar{\lambda}))$ . We can write

$$\mathcal{U} := \Lambda G_\sigma = \mathcal{U}_1 \cup \mathcal{U}_{-1},$$

where the  $\varepsilon$  in  $\mathcal{U}_\varepsilon$  corresponds to that in (2.1). We also have  $\mathcal{U}_1 = \Lambda SU_{1,1}$  and  $\mathcal{U}_{-1} = \begin{pmatrix} 0 & \lambda i \\ \lambda^{-1} i & 0 \end{pmatrix} \cdot \mathcal{U}_1$ . The Lie algebra,  $\text{Lie}(\mathcal{U}) = \text{Lie}(\mathcal{U}_1)$ , of  $\mathcal{U}$ , consists of loops of matrices with analogous properties to those in  $\mathcal{U}$ , replacing the determinant 1 condition with the trace zero condition.

The complexification of  $\mathcal{U}$  is  $\mathcal{U}^\mathbb{C} := \Lambda SL(2, \mathbb{C})_\sigma$ , the group of loops in  $SL(2, \mathbb{C})$  which again have the twisted condition on diagonal/off-diagonal elements mentioned above. Let  $\mathbb{D}_\pm := \{\lambda \in \mathbb{C} \cup \{\infty\} \mid |\lambda|^{\pm 1} < 1\}$ . Three subgroups of  $\mathcal{U}^\mathbb{C}$  that we also use are:

$$\begin{aligned} \mathcal{U}_\pm^\mathbb{C} &:= \{\hat{B} \in \mathcal{U}^\mathbb{C} \mid \hat{B} \text{ extends holomorphically to } \mathbb{D}_\pm\}, \\ \widehat{\mathcal{U}}_+^\mathbb{C} &:= \{\hat{B} \in \mathcal{U}_+^\mathbb{C} \mid \hat{B}|_{\lambda=0} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}, \rho \in \mathbb{R}, \rho > 0\}. \end{aligned}$$

Let  $\Sigma$  be a simply connected non-compact Riemann surface, and suppose  $f : \Sigma \rightarrow \mathbb{L}^3$  is a conformal spacelike immersion with constant mean curvature  $H \neq 0$ , or an  $H$ -surface. Without loss of generality, we assume that  $H > 0$ , the sign being a matter of orientation. If  $z = x + iy$  is a local coordinate, there is a function  $u : \Sigma \rightarrow \mathbb{R}$  such that the metric is given by  $ds^2 =$

$4e^{2u}(dx^2 + dy^2)$ . The *coordinate frame*  $F : \Sigma \rightarrow SU_{1,1}$  is well defined up to premultiplication by  $\pm I$ , by

$$(2.2) \quad Fe_1 F^{-1} = \frac{f_x}{|f_x|}, \quad Fe_2 F^{-1} = \frac{f_y}{|f_y|}.$$

Choose the conformal coordinates  $x$  and  $y$  such that the oriented unit normal is then given by  $N = Fe_3 F^{-1}$ . The Hopf differential is defined to be  $Qdz^2$ , where  $Q := \langle N, f_{zz} \rangle = -\langle N_z, f_z \rangle$ . The Maurer-Cartan form,  $\alpha$ , for the frame  $F$  is defined to be  $\alpha := F^{-1}dF = Udz + Vd\bar{z}$ , where the connection coefficients  $U := F^{-1}F_z$  and  $V := F^{-1}F_{\bar{z}}$  are given by

$$(2.3) \quad U = \frac{1}{2} \begin{pmatrix} u_z & -2iHe^u \\ ie^{-u}Q & -u_z \end{pmatrix}, \quad V = \frac{1}{2} \begin{pmatrix} -u_{\bar{z}} & -ie^{-u}\bar{Q} \\ 2iHe^u & u_{\bar{z}} \end{pmatrix}.$$

The compatibility condition  $d\alpha + \alpha \wedge \alpha = 0$  is equivalent to the pair of equations

$$(2.4) \quad u_{z\bar{z}} - H^2 e^{2u} + \frac{1}{4}|Q|^2 e^{-2u} = 0, \quad Q_{\bar{z}} = 2e^{2u}H_z.$$

The above structure for  $U$  and  $V$  are verified by a computation, using  $H = \frac{1}{8}e^{-2u}\langle f_{xx} + f_{yy}, N \rangle$ , and

$$(2.5) \quad f_z = 2e^u F \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot F^{-1}, \quad f_{\bar{z}} = 2e^u F \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot F^{-1}.$$

We can insert an  $\mathbb{S}^1$  parameter  $\lambda$  into the 1-form  $\alpha$ , defining a family  $\hat{\alpha} := \hat{U}dz + \hat{V}d\bar{z}$ , where

$$(2.6) \quad \hat{U} = \frac{1}{2} \begin{pmatrix} u_z & -2iHe^u\lambda^{-1} \\ ie^{-u}Q\lambda^{-1} & -u_z \end{pmatrix}, \quad \hat{V} = \frac{1}{2} \begin{pmatrix} -u_{\bar{z}} & -ie^{-u}\bar{Q}\lambda \\ 2iHe^u\lambda & u_{\bar{z}} \end{pmatrix}.$$

Then the assumption that  $H$  is constant is equivalent to the integrability of  $\hat{\alpha}$  for all  $\lambda$ . Hence it can be integrated to obtain a map  $\hat{F} : \Sigma \rightarrow \mathcal{U}_1$ . Supposing that our coordinate frame  $F$  defined above satisfies  $F(z_0) = F_0$ , at some point  $z_0$ , we integrate  $\hat{\alpha}$  with the same initial condition, and call the map  $\hat{F} : \Sigma \rightarrow \mathcal{U}_1$  thus obtained an *extended frame* for the  $H$ -surface  $f$ .

The Sym-Bobenko formula is the map  $\mathcal{S} : \mathcal{U} \rightarrow \text{Lie}(\mathcal{U})$  given by:

$$(2.7) \quad \mathcal{S}(\hat{F}) := -\frac{1}{2H} \left( \hat{F}e_3\hat{F}^{-1} + 2i\lambda \frac{\partial \hat{F}}{\partial \lambda} \hat{F}^{-1} \right).$$

We write  $\mathcal{S}_\lambda : \mathcal{U} \rightarrow \mathbb{L}^3$  for the map given by evaluating this at  $\lambda \in \mathbb{S}^1$ . If  $\hat{F} : \Sigma \rightarrow \mathcal{U}_1$  is an extended frame for an  $H$ -surface  $f$ , then, up a translation in  $\mathbb{L}^3$ , the surface is retrieved by applying the Sym-Bobenko formula at  $\lambda = 1$ :

$$f = \mathcal{S}_1(\hat{F}) + \text{translation}.$$

This is verified by computing  $\mathcal{S}_1(\hat{F})_z$  and  $\mathcal{S}_1(\hat{F})_{\bar{z}}$ , using the matrices  $\hat{U}$  and  $\hat{V}$ . The same computation shows that  $\mathcal{S}_{\lambda_0}(\hat{F})$  is also an  $H$ -surface for any  $\lambda_0 \in \mathbb{S}^1$ . For such computations, note that if

$$\hat{G}^{-1}\hat{G}_z = \begin{pmatrix} u_0 & \alpha\lambda^{-1} \\ \beta\lambda^{-1} & -u_0 \end{pmatrix}, \quad \hat{G}^{-1}\hat{G}_{\bar{z}} = \begin{pmatrix} -\bar{u}_0 & \bar{\beta}\lambda \\ \bar{\alpha}\lambda & \bar{u}_0 \end{pmatrix},$$

and we set  $f^\lambda = \mathcal{S}_\lambda(\hat{G})$ , then one computes the following formulae:

$$(2.8) \quad \hat{G}^{-1}f_z^\lambda \hat{G} = \frac{2i}{H} \begin{pmatrix} 0 & \alpha\lambda^{-1} \\ 0 & 0 \end{pmatrix}, \quad \hat{G}^{-1}f_{\bar{z}}^\lambda \hat{G} = \frac{2i}{H} \begin{pmatrix} 0 & 0 \\ -\bar{\alpha}\lambda & 0 \end{pmatrix}.$$

One can also define a CMC surface with extended coordinate frame  $\tilde{F}$  in the other half of the loop group,  $\mathcal{U}_{-1}$ , by integrating the 1-form  $\hat{U}dz + \hat{V}d\bar{z}$  with the initial condition

$$\tilde{F}(z_0) = W = \begin{pmatrix} 0 & i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix}.$$

Since  $\mathcal{S}(W\hat{F}) = \text{Ad}_W \mathcal{S}(\hat{F}) + \text{translation}$  – where  $\text{Ad}_X$  denotes conjugation by  $X$  – and  $\text{Ad}_W$  is an isometry of  $\mathbb{L}^3$ , this is also a CMC surface. If  $\hat{F}$  is the frame obtained with the initial condition  $\hat{F}(z_0) = I$ , then the relation between the surfaces obtained at  $\lambda = 1$  is  $\mathcal{S}_1(\tilde{F}) = \text{Ad}_W|_{\lambda=1} \mathcal{S}_1(\hat{F}) + \text{translation}$ . The coordinate frame for  $\tilde{f} = \mathcal{S}_1(\tilde{F})$  satisfies  $\tilde{F}e_1\tilde{F}|_{\lambda=1} = \frac{\tilde{f}_x}{|\tilde{f}_x|}$  and  $\tilde{F}e_2\tilde{F}|_{\lambda=1} = \frac{\tilde{f}_y}{|\tilde{f}_y|}$ .

More generally, one can show (see, for example, the analogous argument in [4]):

**Lemma 2.1.** *If  $\hat{F} : \Sigma \rightarrow \mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_{-1}$  is a real analytic map the Maurer-Cartan form of which has the form*

$$(2.9) \quad \hat{F}^{-1}d\hat{F} = \alpha_{-1}dz\lambda^{-1} + \hat{\beta}dz + \hat{\gamma}d\bar{z},$$

where the loop-algebra valued functions  $\hat{\beta}$  and  $\hat{\gamma}$  extend holomorphically in  $\lambda$  to the unit disc, and with the regularity condition  $[\alpha_{-1}]_{12} \neq 0$ , then the map  $f^{\lambda_0} = \mathcal{S}_{\lambda_0}(\hat{F})$  is an  $H$ -surface in  $\mathbb{L}^3$ , and the coordinate frame for this surface is given by  $F = \hat{F}|_{\lambda_0}D$ , where  $D : \Sigma \rightarrow G$  is a diagonal matrix-valued function.

Note that the Sym-Bobenko formula is invariant under gauge transformations  $\hat{F} \mapsto \hat{F}D$ , where  $D$  is constant in  $\lambda$  and diagonal. It also follows from the fact that the 1-form  $\hat{F}^{-1}d\hat{F}$  of Lemma 2.1 takes values in  $\text{Lie}(\mathcal{U})$  that, in fact,

$$\hat{F}^{-1}d\hat{F} = \alpha_{-1}dz\lambda^{-1} + \alpha_0dz + \tau(\alpha_0)d\bar{z} + \tau(\alpha_{-1})d\bar{z}\lambda,$$

where the involution  $\tau$  that defines  $\mathfrak{g} = \mathfrak{su}_{1,1}$  as a real form of  $\mathfrak{sl}(2, \mathbb{C})$  is given by:

$$\tau(X) := -\text{Ad}_\sigma X^t, \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

**2.2. Construction of solutions via the DPW method.** By Lemma 2.1, the problem of constructing a conformal spacelike CMC immersion  $f : \Sigma \rightarrow \mathbb{L}^3$  is evidently equivalent to the problem of constructing a real analytic map  $\hat{F} : \Sigma \rightarrow \mathcal{U}$ , such that  $\hat{F}^{-1}d\hat{F}$  is of the type given by (2.9). The DPW construction does exactly that, beginning with an arbitrary holomorphic map  $\hat{\Phi} : \Sigma \rightarrow \mathcal{U}^\mathbb{C}$  which satisfies  $\hat{\Phi}^{-1}d\hat{\Phi} = (\beta_{-1}\lambda^{-1} + \beta_0 + \dots)dz$ .

In order to explain this, we first need to state the Iwasawa decomposition of  $\mathcal{U}^\mathbb{C}$ . Define, for a positive integer  $m \in \mathbb{Z}^+$ ,

$$\omega_m = \begin{pmatrix} 1 & 0 \\ \lambda^{-m} & 1 \end{pmatrix}, \quad m \text{ odd}; \quad \omega_m = \begin{pmatrix} 1 & \lambda^{1-m} \\ 0 & 1 \end{pmatrix}, \quad m \text{ even}.$$

**Theorem 2.2.** ( $SU_{1,1}$  Iwasawa decomposition [5])

(1) *The group  $\mathcal{U}^\mathbb{C}$  is a disjoint union*

$$(2.10) \quad \mathcal{U}^\mathbb{C} = \mathcal{B}_{1,1} \sqcup \bigsqcup_{m \in \mathbb{Z}^+} \mathcal{P}_m,$$

where

$$\mathcal{B}_{1,1} := \mathcal{U} \cdot \mathcal{U}_+^\mathbb{C},$$

is called the big cell, and the  $n$ -th small cell is:

$$(2.11) \quad \mathcal{P}_n := \mathcal{U}_1 \cdot \omega_n \cdot \mathcal{U}_+^\mathbb{C}.$$

(2) *In the factorization*

$$(2.12) \quad \hat{\Phi} = \hat{F}\hat{B}, \quad \hat{F} \in \mathcal{U}, \quad \hat{B} \in \mathcal{U}_+^{\mathbb{C}},$$

of a loop  $\hat{\Phi} \in \mathcal{B}_{1,1}$ , the factor  $\hat{F}$  is unique up to right multiplication by an element of the subgroup  $\mathcal{U}^0$  of constant loops in  $\mathcal{U}$ . Both factors are unique if we require that  $\hat{B} \in \widehat{\mathcal{U}}_+^{\mathbb{C}}$ , and with this normalization the product map  $\mathcal{U} \times \widehat{\mathcal{U}}_+^{\mathbb{C}} \rightarrow \mathcal{B}_{1,1}$  is a real analytic diffeomorphism.

(3) *The Iwasawa big cell,  $\mathcal{B}_{1,1}$ , is an open dense subset of  $\mathcal{U}^{\mathbb{C}}$ . The complement of  $\mathcal{B}_{1,1}$  in  $\mathcal{U}^{\mathbb{C}}$  is locally given as the zero set of a non-constant real analytic function  $\mathcal{U}^{\mathbb{C}} \rightarrow \mathbb{C}$ .*

It is clear from Theorem 2.2 that the big cell  $\mathcal{B}_{1,1}$  is naturally divided into two disjoint open sets corresponding to whether the element  $\hat{F}$  is a loop in  $SU_{1,1}$  or in  $ie_1SU_{1,1}$ . We denote these subsets by  $\mathcal{B}_{1,1}^+$  and  $\mathcal{B}_{1,1}^-$  respectively.

Now it is easy to check that if  $\hat{\Phi} : \Sigma \rightarrow \mathcal{B}_{1,1} \subset \mathcal{U}^{\mathbb{C}}$  satisfies  $\hat{\Phi}^{-1}d\hat{\Phi} = (\beta_{-1}\lambda^{-1} + \beta_0 + \dots)dz$ , and  $\hat{\Phi} = \hat{F}\hat{B}$  is an Iwasawa factorization of  $\hat{\Phi}$ , with  $\hat{F} \in \mathcal{U}$ , then  $\hat{F}^{-1}d\hat{F}$  is of the required form (2.9). That is the essential point behind the generalized Weierstrass representation for  $H$ -surfaces which will be stated in the next theorem.

**Definition 2.3.** A standard (holomorphic) potential on a Riemann surface  $\Sigma$  is a holomorphic 1-form  $\hat{\xi} \in \text{Lie}(\mathcal{U}^{\mathbb{C}}) \otimes \Omega^{1,0}(\Sigma)$ , the Fourier expansion of which begins at  $\lambda^{-1}$ :

$$\hat{\xi} = \sum_{i=-1}^{\infty} \beta_i \lambda^i dz, \quad \beta_i : \Sigma \rightarrow \mathfrak{sl}(2, \mathbb{C}), \text{ holomorphic,}$$

and with the regularity condition on the (1,2) component of  $\beta_{-1}$ :

$$[\beta_{-1}]_{12}(z) \neq 0, \quad \forall z \in \Sigma.$$

**Theorem 2.4.** [5]. Let  $\hat{\xi}$  be a standard holomorphic potential on a simply-connected Riemann surface  $\Sigma$ . Let  $\hat{\Phi} : \Sigma \rightarrow \mathcal{U}^{\mathbb{C}}$  be a solution of

$$\hat{\Phi}^{-1}d\hat{\Phi} = \hat{\xi}.$$

Define the open set  $\Sigma^{\circ} := \hat{\Phi}^{-1}(\mathcal{B}_{1,1})$ . Assume that the map  $\hat{\Phi}$ , maps at least one point into  $\mathcal{B}_{1,1}$ , so that  $\Sigma^{\circ}$  is not empty, and take any  $G$ -Iwasawa splitting pointwise on  $\Sigma^{\circ}$ :

$$(2.13) \quad \hat{\Phi} = \hat{F}\hat{B}, \quad \hat{F} \in \mathcal{U}, \quad \hat{B} \in \mathcal{U}_+^{\mathbb{C}}.$$

Then for any  $\lambda_0 \in \mathbb{S}^1$ , the map  $f^{\lambda_0} := \mathcal{S}_{\lambda_0}(\hat{F}) : \Sigma^{\circ} \rightarrow \mathbb{L}^3$ , given by the Sym-Bobenko formula (2.7), is a conformal spacelike CMC  $H$  immersion, and is independent of the choice of  $\hat{F} \in \mathcal{U}$  in (2.13).

Conversely, let  $\Sigma$  be a noncompact Riemann surface. Then any non-maximal conformal CMC spacelike immersion from  $\Sigma$  into  $\mathbb{L}^3$  can be constructed in this manner, using a holomorphic potential  $\hat{\xi}$  that is well-defined on  $\Sigma$ .

We call  $\hat{\Phi}$  a *holomorphic extended frame* for the family of surfaces  $f^{\lambda}$ . It is also true that if we normalize the factors in (2.13) so that  $\hat{B} \in \widehat{\mathcal{U}}_+^{\mathbb{C}}$ , and define the function  $\rho : \Sigma^{\circ} \rightarrow \mathbb{R}$  by  $\hat{B}|_{\lambda=0} = \text{diag}(\rho, \rho^{-1})$ , then there exist conformal coordinates  $\tilde{z} = \tilde{x} + i\tilde{y}$  on  $\Sigma$  such that the induced metric for  $f^1$  is given by

$$ds^2 = 4\rho^4(d\tilde{x}^2 + d\tilde{y}^2),$$

and the Hopf differential is given by  $Qd\tilde{z}^2$ , where  $Q = -2H \frac{b_{-1}}{a_{-1}}$ .



**2.3. Behaviour of the surface at the boundary of the big cell.** Theorem 2.4 says that a standard holomorphic potential  $\hat{\xi}$  corresponds to an  $H$ -surface, provided we restrict to  $\Sigma^\circ = \hat{\Phi}^{-1}(\mathcal{B}_{1,1})$ . Now set

$$\mathcal{C} := \Sigma \setminus \Sigma^\circ = \bigcup_{j=1}^{\infty} \hat{\Phi}^{-1}(\mathcal{P}_j), \quad \mathcal{C}_1 := \hat{\Phi}^{-1}(\mathcal{P}_1), \quad \mathcal{C}_2 := \hat{\Phi}^{-1}(\mathcal{P}_2).$$

**Theorem 2.5.** [5] *Let  $\hat{\Phi}$  be as defined in Theorem 2.4. Then*

- (1)  $\Sigma^\circ$  is open and dense in  $\Sigma$ . More precisely, its complement, the set  $\mathcal{C}$ , is locally given as the zero set of a non-constant real analytic function  $\Sigma \rightarrow \mathbb{C}$ .
- (2) The sets  $\Sigma^\circ \cup \mathcal{C}_1$  and  $\Sigma^\circ \cup \mathcal{C}_2$  are both open subsets of  $\Sigma$ . The sets  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are each locally given as the zero set of a non-constant real analytic function  $\Sigma \rightarrow \mathbb{R}$ .
- (3) All components of any matrix  $F$  obtained by Theorem 2.4 on  $\Sigma^\circ$ , and evaluated at  $\lambda_0 \in \mathbb{S}^1$ , blow up as  $z$  approaches a point  $z_0$  in either  $\mathcal{C}_1$  or  $\mathcal{C}_2$ . In the limit, the unit normal vector  $N$ , to the corresponding surface, becomes asymptotically lightlike, i.e. its length in the Euclidean space  $\mathbb{R}^3$  metric approaches infinity.
- (4) The surface  $f^{\lambda_0}$  obtained from Theorem 2.4 extends to a real analytic map  $\Sigma^\circ \cup \mathcal{C}_1 \rightarrow \mathbb{L}^3$ , but is not immersed at points  $z_0 \in \mathcal{C}_1$ .
- (5) The surface  $f^{\lambda_0}$  diverges to  $\infty$  as  $z \rightarrow z_0 \in \mathcal{C}_2$ . Moreover, the induced metric on the surface blows up as such a point in the coordinate domain is approached.

The arguments given in [5] to prove those parts of the above theorem involving  $\mathcal{C}_1$  and  $\mathcal{C}_2$  all depend on an explicit Iwasawa factorization of an element of the form  $B\omega_1$ , where  $B$  is an arbitrary element of  $\mathcal{W}_+^{\mathbb{C}}$ . We will use this explicit factorization again several times below, and so we recall it here:

**Lemma 2.6.** [5] *Let  $\hat{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \mu \\ v & 0 \end{pmatrix} \lambda + o(\lambda^2)$  be any element of  $\mathcal{W}_+^{\mathbb{C}}$ . Then there exists a factorization*

$$(2.14) \quad \hat{B}\omega_1 = \hat{X}\hat{B}',$$

where  $\hat{B}' \in \mathcal{W}_+^{\mathbb{C}}$  and  $\hat{X}$  is of one of the following three forms:

$$k_1 = \begin{pmatrix} u & v\lambda \\ \bar{v}\lambda^{-1} & \bar{u} \end{pmatrix}, \quad k_2 = \begin{pmatrix} u & v\lambda \\ -\bar{v}\lambda^{-1} & -\bar{u} \end{pmatrix}, \quad \omega_1^\theta = \begin{pmatrix} 1 & 0 \\ e^{i\theta}\lambda^{-1} & 1 \end{pmatrix},$$

where  $u$  and  $v$  are constant in  $\lambda$  and can be chosen so that the matrix has determinant one, and  $\theta \in \mathbb{R}$ . The matrices  $k_1$  and  $k_2$  are in  $\mathcal{W}$ , and their components satisfy the equation

$$(2.15) \quad \frac{|u|}{|v|} = |\mu + \rho||\rho|.$$

The first two forms occurs when  $\hat{B}\omega_1$  is in the big cell  $\mathcal{B}_{1,1}$ , and the third form occurs if and only if  $\hat{B}\omega_1$  is in the first small cell,  $\mathcal{P}_1$ . The three cases correspond to the cases  $|(\mu + \rho)\rho|$  greater than, less than or equal to 1, respectively. Moreover, if  $\hat{B}\omega_1$  is given locally by a real analytic map either from  $\mathbb{R}^2 \rightarrow \mathcal{B}_{1,1}$ , or from  $\mathbb{R} \rightarrow \mathcal{P}_1$ , then the factors  $\hat{X}$  and  $\hat{B}'$  can be chosen to be real analytic.

*Proof.* One can write down explicit expressions as follows: for the cases  $|(\mu + \rho)\rho|^\varepsilon > 1$ , where  $\varepsilon = \pm 1$ , the factorization is given by

$$(2.16) \quad \begin{aligned} \hat{X} &= \begin{pmatrix} u & v\lambda \\ \varepsilon \bar{v}\lambda^{-1} & \varepsilon \bar{u} \end{pmatrix}, \\ \hat{B}' &= \begin{pmatrix} \varepsilon \bar{u}b\lambda^{-1} - dv + \varepsilon \bar{u}a - vc\lambda & b\varepsilon \bar{u} - vd\lambda \\ -\varepsilon \bar{v}b\lambda^{-2} + (-\varepsilon \bar{v}a + ud)\lambda^{-1} + uc & -b\varepsilon \bar{v}\lambda^{-1} + ud \end{pmatrix}. \end{aligned}$$

One can choose  $u$  and  $v$  so that  $\varepsilon(u\bar{u} - v\bar{v}) = 1$  and such that  $\hat{B}' \in \mathcal{U}_+^{\mathbb{C}}$ , the latter condition being assured by the requirement that  $\frac{u}{\bar{v}} = \varepsilon(\mu + \rho)\rho$ . Once such choice is

$$(2.17) \quad v = \frac{1}{\sqrt{\varepsilon(|\mu + \rho|^2 |\rho|^2 - 1)}}, \quad u = \varepsilon(\mu + \rho)\rho \bar{v}.$$

It is straightforward to verify that  $\hat{X}\hat{B}' = \hat{B}\omega_1^{-1}$ .

For the case  $|(\mu + \rho)\rho| = 1$ , use

$$(2.18) \quad \begin{aligned} \hat{X} &= \begin{pmatrix} u & v\lambda \\ -\bar{v}\lambda^{-1} & \bar{u} \end{pmatrix}, \\ \hat{B}' &= \begin{pmatrix} \bar{u}b\lambda^{-1} - dv + \bar{u}a - vc\lambda & b\bar{u} - vd\lambda \\ \bar{v}b\lambda^{-2} + (\bar{v}a + ud)\lambda^{-1} + uc & b\bar{v}\lambda^{-1} + ud \end{pmatrix}. \end{aligned}$$

and choose  $\frac{u}{\bar{v}} = -(\mu + \rho)\rho$ . One can choose  $u = \frac{1}{\sqrt{2}}$  and  $\bar{v} = \frac{-1}{\sqrt{2}}((\mu + \rho)\rho)^{-1} = \frac{-1}{\sqrt{2}}e^{i\theta}$  and

$$\begin{pmatrix} u & v\lambda \\ -\bar{v}\lambda^{-1} & \bar{u} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{i\theta}\lambda^{-1} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}e^{-i\theta}\lambda \\ 0 & \sqrt{2} \end{pmatrix}.$$

Pushing the last factor into  $\hat{B}'$  then gives the required factorization. In this case,  $\hat{B}\omega_1^{-1}$  is in  $\mathcal{P}_1$ , because it can be expressed as

$$\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \cdot \omega_1 \cdot \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \hat{B}'.$$

The claimed analytic properties of the factors are satisfied for the explicit choices of  $u$  and  $v$  given above, because the expression  $(\mu + \rho)\rho$  is real analytic.  $\square$

### 3. THE WEIERSTRASS REPRESENTATION FOR SURFACES WITH SINGULARITIES

Theorem 2.5 states that singularities occur at points which are mapped into  $\mathcal{P}_1$ , and that the frame  $F$  is not defined at such points. In this section we define an alternative extended frame  $\hat{F}_\omega$  which does not blow up at singular points. This will be used in the next section to solve the singular Björling problem.

Let  $\pi : \mathcal{B}_{1,1} \rightarrow \mathcal{U}/\mathcal{U}^0$  denote the projection defined by taking the equivalence class of  $\hat{F}$  (under right multiplication by elements of  $\mathcal{U}^0$ ) in the Iwasawa factorization  $\hat{\Phi} = \hat{F}\hat{B}$  of  $\hat{\Phi} \in \mathcal{B}_{1,1}$ . Since the Sym-Bobenko formula  $\mathcal{S}$  is invariant under right multiplication by constant diagonal matrices,  $\mathcal{S} : \mathcal{U}/\mathcal{U}^0 \rightarrow \text{Lie}(\mathcal{U})$  is well defined, and we can extend it to a map

$$\widetilde{\mathcal{S}} : \mathcal{B}_{1,1} \rightarrow \text{Lie}(\mathcal{U}), \quad \widetilde{\mathcal{S}} = \mathcal{S} \circ \pi.$$

Again we define the map  $\widetilde{\mathcal{S}}_\lambda : \mathcal{B}_{1,1} \rightarrow \mathbb{L}^3$  by evaluating this at  $\lambda \in \mathbb{S}^1$ . The crucial fact that is exploited here and in [5] – and is proved using Lemma 2.6 – is that if  $\hat{\Phi} \in \mathcal{B}_{1,1}$  and  $\hat{\Phi}\omega_1^{-1} \in \mathcal{B}_{1,1}$  then

$$(3.1) \quad \widetilde{\mathcal{S}}(\hat{\Phi}\omega_1^{-1}) = \widetilde{\mathcal{S}}(\hat{\Phi}).$$

Thus, if  $\hat{\Phi} : \Sigma \rightarrow \mathcal{U}^{\mathbb{C}}$ , and  $\hat{\Phi}(z_0) = \omega_1 \in \mathcal{P}_1$ , then we can just as well consider the map  $\hat{\Phi}_\omega := \hat{\Phi}\omega_1^{-1}$ . Then  $\hat{\Phi}_\omega(z) \in \mathcal{B}_{1,1}$  in a neighbourhood of  $z_0$ , and if  $\hat{\Phi}$  is a holomorphic extended frame, then so is  $\hat{\Phi}_\omega$  – for the same family of surfaces  $f^\lambda$ . On the open dense set  $\hat{\Phi}^{-1}(\mathcal{B}_{1,1}) \cap \hat{\Phi}_\omega^{-1}(\mathcal{B}_{1,1})$ , we have  $\widetilde{\mathcal{F}}(\hat{\Phi}) = \widetilde{\mathcal{F}}(\hat{\Phi}_\omega)$ , and so it is valid to define

$$f^{\lambda_0}(z_0) := \widetilde{\mathcal{F}}_{\lambda_0}(\hat{\Phi}_\omega(z_0)).$$

Any element of  $\mathcal{P}_1$  is of the form  $\hat{F}_0\omega_1\hat{B}_0$ , and essentially the same argument can be used to define  $f^{\lambda_0}(z_0)$  when  $\hat{\Phi}(z_0)$  has this form. Hence one can define a real analytic map  $f^{\lambda_0} : \hat{\Phi}^{-1}(\mathcal{B}_{1,1} \cup \mathcal{P}_1) \rightarrow \mathbb{L}^3$  which is an immersed CMC  $H$  surface on  $\hat{\Phi}^{-1}(\mathcal{B}_{1,1})$ .

**Definition 3.1.** Let  $\Sigma$  be a simply-connected Riemann surface,  $\hat{\xi}$  a standard potential, and  $\hat{\Phi} : \Sigma \rightarrow \mathcal{U}^{\mathbb{C}}$  the map obtained by integrating  $\hat{\Phi}^{-1}d\hat{\Phi} = \hat{\xi}$  with an initial condition  $\hat{\Phi}(z_0) = \hat{\Phi}_0 \in \mathcal{U}^{\mathbb{C}}$ . Assume that  $\hat{\Phi}(w) \in \mathcal{B}_{1,1}$  for at least one point  $w \in \Sigma$ . Let  $\Sigma_s \subset \Sigma$  be the open dense subset given by  $\Sigma_s = \hat{\Phi}^{-1}(\mathcal{B}_{1,1} \cup \mathcal{P}_1)$ , and define, for any  $\lambda \in \mathbb{S}^1$ ,

$$f^\lambda : \Sigma_s \rightarrow \mathbb{L}^3, \quad f^\lambda(z) = \widetilde{\mathcal{F}}_\lambda(\hat{\Phi}(z)).$$

We call the map  $f^\lambda$  – and, more generally, any map from a Riemann surface into  $\mathbb{L}^3$  which has such a representation locally – a generalized constant mean curvature  $H$  surface, or generalized  $H$ -surface, in  $\mathbb{L}^3$ .

**3.1. Singular holomorphic potentials and frames.** For a typical generalized  $H$ -surface we can expect, from Theorem 2.5 Item 2, that the singular set  $\mathcal{C}_1 = \hat{\Phi}^{-1}(\mathcal{P}_1)$  is a curve, and we can deduce from Item 3 that this curve must be a null curve, wherever it is regular.

It is clear from the preceding discussion that one may construct a generalized  $H$ -surface with a singularity at  $z_0$  by integrating a standard potential  $\hat{\xi}$  with the initial condition  $\hat{\Phi}(z_0) = \omega_1$ , provided that the resulting complex extended frame  $\hat{\Phi}$  does satisfy  $\hat{\Phi}(z) \in \mathcal{B}_{1,1}$  for some  $z$ . Alternatively, supposing we did this, there is also the translated map  $\hat{\Phi}_\omega = \hat{\Phi}\omega_1^{-1}$  – which may be more natural because  $\hat{\Phi}_\omega(z_0) = I$  and so this maps a neighbourhood of  $z_0$  into the big cell.

We first analyze the Maurer-Cartan form of  $\hat{\Phi}_\omega$ , given that  $\hat{\xi}$  is a standard potential, which has the general form:

$$(3.2) \quad \hat{\Phi}^{-1}d\hat{\Phi} = \left\{ \begin{pmatrix} 0 & a_{-1} \\ b_{-1} & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} c_0 & 0 \\ 0 & -c_0 \end{pmatrix} + \begin{pmatrix} 0 & a_1 \\ b_1 & 0 \end{pmatrix} \lambda + o(\lambda^2) \right\} dz,$$

where  $a_{-1}$  is non-vanishing. For  $\hat{\Phi}_\omega = \hat{\Phi}\omega_1^{-1}$ , the above expression is equivalent to

$$\begin{aligned} \hat{\Phi}_\omega^{-1}d\hat{\Phi}_\omega &= \left\{ \begin{pmatrix} 0 & 0 \\ -a_{-1} & 0 \end{pmatrix} \lambda^{-3} + \begin{pmatrix} -a_{-1} & 0 \\ 0 & a_{-1} \end{pmatrix} \lambda^{-2} + \begin{pmatrix} 0 & a_{-1} \\ b_{-1} + 2c_0 - a_1 & 0 \end{pmatrix} \lambda^{-1} \right. \\ &\quad \left. + \begin{pmatrix} c_0 - a_1 & 0 \\ 0 & -c_0 + a_1 \end{pmatrix} + \begin{pmatrix} 0 & a_1 \\ b_1 & 0 \end{pmatrix} \lambda + o(\lambda^2) \right\} dz. \end{aligned}$$

Now consider the special case that  $\hat{\Phi}_\omega(z) \in \mathcal{U}$  for  $z \in \mathbb{R}$ . Then the Iwasawa factorization of  $\hat{\Phi}_\omega$  along  $\mathbb{R}$ , is just  $\hat{\Phi}_\omega = \hat{\Phi}_\omega \cdot I$ , and therefore the Iwasawa factorization of  $\hat{\Phi}$  for  $z \in \mathbb{R}$  is just  $\hat{\Phi} = \hat{\Phi}_\omega \cdot \omega_1 \cdot I$ . In other words, such a holomorphic frame maps the real line into  $\mathcal{P}_1$ . The assumption is equivalent to demanding that  $\hat{\Phi}_\omega^{-1} \frac{\partial \hat{\Phi}_\omega}{\partial x}(x, 0) dx$  has coefficients in  $\text{Lie}(\mathcal{U})$ ,

which implies that it must be of the form:

$$(3.3) \quad \hat{\xi}_0 = \left\{ \begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix} \lambda^{-3} + \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} \lambda^{-2} + \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} ir & 0 \\ 0 & -ir \end{pmatrix} \right. \\ \left. + \begin{pmatrix} 0 & \bar{b} \\ \bar{a} & 0 \end{pmatrix} \lambda + \begin{pmatrix} \bar{a} & 0 \\ 0 & -\bar{a} \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & -\bar{a} \\ 0 & 0 \end{pmatrix} \lambda^3 \right\} dx,$$

where  $a$  and  $b$  are maps  $\mathbb{R} \rightarrow \mathbb{C}$  while  $r : \mathbb{R} \rightarrow \mathbb{R}$ , and all functions are restrictions to  $\mathbb{R}$  of holomorphic functions. Hence, the Maurer-Cartan form of  $\hat{\Phi}_\omega$  is a holomorphic extension of this:

**Definition 3.2.** Let  $\Sigma \subset \mathbb{C}$  be a simply connected open subset which intersects the real line in an interval:  $\Sigma \cap \mathbb{R} = J = (x_0, x_1)$ , and contains the origin  $z = 0$ . A standard singular holomorphic potential on  $\Sigma$ , is a holomorphic 1-form  $\hat{\xi}_\omega$  on  $\Sigma$  that can be expressed as:

$$(3.4) \quad \hat{\xi}_\omega = \hat{\Phi}_\omega^{-1} d\hat{\Phi}_\omega = \left\{ \begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix} \lambda^{-3} + \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} \lambda^{-2} + \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} ir & 0 \\ 0 & -ir \end{pmatrix} \right. \\ \left. + \begin{pmatrix} 0 & \bar{b} \\ \bar{a} & 0 \end{pmatrix} \lambda + \begin{pmatrix} \bar{a} & 0 \\ 0 & -\bar{a} \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & -\bar{a} \\ 0 & 0 \end{pmatrix} \lambda^3 \right\} dz,$$

where  $a, b$  and  $r$  are holomorphic on  $\Sigma$ , the restriction of  $r$  to  $J$  is real, that is  $\overline{r(\bar{z})} = r(z)$ , and  $\bar{a}$  and  $\bar{b}$  are holomorphic extensions of the restrictions  $\bar{a}|_{\mathbb{R}}$  and  $\bar{b}|_{\mathbb{R}}$ , that is  $\bar{a}(z) = \overline{a(\bar{z})}$ , and  $\bar{b}(z) = \overline{b(\bar{z})}$ , with the regularity condition:

(A)  $a(z)$  non-vanishing on  $\Sigma$ .

Define the singular holomorphic frame  $\hat{\Phi}_\omega$  corresponding to  $\hat{\xi}_\omega$  to be the map  $\hat{\Phi}_\omega : \Sigma \rightarrow \mathcal{U}^{\mathbb{C}}$  obtained by solving the equation

$$\hat{\Phi}_\omega^{-1} d\hat{\Phi}_\omega = \hat{\xi}_\omega, \quad \hat{\Phi}_\omega(0) = I.$$

Set

$$\begin{aligned} \hat{\Phi} &:= \hat{\Phi}_\omega \omega_1, \\ \Sigma^\circ &:= \hat{\Phi}^{-1}(\mathcal{B}_{1,1}), & C &:= \hat{\Phi}^{-1}(\mathcal{P}_1), & \Sigma_s &:= \Sigma^\circ \cup C. \end{aligned}$$

Note that  $\hat{\Phi}(0) = \omega_1 \notin \mathcal{B}_{1,1}$  so it is not clear that  $\Sigma^\circ$  is non-empty.

**Theorem 3.3.** Suppose  $\hat{\xi}_\omega$  is a standard singular holomorphic potential given by Definition 3.2, and suppose that  $\Sigma^\circ$  is non-empty. Then

- (1)  $\Sigma^\circ$  is open and dense in  $\Sigma$ .
- (2)  $\Sigma_s$  is also an open dense subset of  $\Sigma$ . For any  $\lambda \in \mathbb{S}^1$ , the map  $f^\lambda : \Sigma_s \rightarrow \mathbb{L}^3$ , given by

$$\begin{aligned} f^\lambda &= \widetilde{\mathcal{F}}_\lambda(\hat{\Phi}_\omega) \\ &= \widetilde{\mathcal{F}}_\lambda(\hat{\Phi}), \end{aligned}$$

is a generalized constant mean curvature  $H$  surface.

- (3) The restriction  $f^\lambda|_{\Sigma^\circ} : \Sigma^\circ \rightarrow \mathbb{L}^3$  is a spacelike CMC  $H$  immersion.
- (4) The map  $f^\lambda$  is not immersed at points  $z \in C$ , and the interval  $J = \Sigma \cap \mathbb{R}$  is contained in the singular set  $C$ . Moreover,  $f^\lambda|_J$  is either a single point or a real analytic null curve which is regular except at points where  $\operatorname{Re}(a\lambda^{-2}) = 0$ .
- (5) A condition that ensures that  $\Sigma^\circ$  is non-empty is:
  - (B)  $r - \operatorname{Im} b$  not equivalent to zero on  $J = \Sigma \cap \mathbb{R}$ .

Moreover, on a neighbourhood in  $\Sigma$  of a point  $z_0 \in J$ , such that  $r(z_0) - \text{Im } b(z_0) \neq 0$ , the sets  $C$  and  $J$  coincide.

*Proof.* **Items 1-3:** The Maurer-Cartan form of  $\hat{\Phi} = \hat{\Phi}_\omega \omega_1$  is given by

$$(3.5) \quad \hat{\Phi}^{-1} d\hat{\Phi} = \begin{pmatrix} ir + \tilde{b} & a\lambda^{-1} + \tilde{b}\lambda - \tilde{a}\lambda^3 \\ 2i(\frac{1}{2i}(b - \tilde{b}) - r)\lambda^{-1} & -ir - \tilde{b} \end{pmatrix} dz,$$

and we assumed  $a$  is non-vanishing, so this is a standard holomorphic potential. Since  $\hat{\xi}_\omega$  is  $\text{Lie}(\mathcal{U})$ -valued along  $\mathbb{R}$ , it follows that  $\hat{\Phi}_\omega$  maps  $J \subset \mathbb{R}$  into  $\mathcal{U}$ . Therefore  $\hat{\Phi} = \hat{\Phi}_\omega \omega_1$  maps  $J$  into  $\mathcal{P}_1$ , by definition of  $\mathcal{P}_1$ . Hence items 1-3 follow from Theorem 2.5 and equation (3.1) above.

**Item 4:** The first statement follows from Theorem 2.5, so we are left with the second statement concerning the regularity of  $f^\lambda|_J$ .

First, since  $\hat{\Phi}_\omega(z) \in \mathcal{U} \subset \mathcal{B}_{1,1}$  for real values of  $z$ , it follows that the set  $W = \hat{\Phi}_\omega^{-1}(\mathcal{B}_{1,1})$  is open (and, in fact dense, see the proof of Theorem 4.1 of [5]) and contains  $J$ . Hence, pointwise on this set, we can decompose

$$\begin{aligned} \hat{\Phi}_\omega &= \hat{F}_\omega \hat{B}_\omega, & \hat{F}_\omega &\in \mathcal{U}, & \hat{B}_\omega &\in \widehat{\mathcal{U}}_+^{\mathbb{C}} \\ \hat{F}_\omega|_J &= \hat{\Phi}_\omega|_J, & \hat{B}_\omega|_J &= I. \end{aligned}$$

We will call  $\hat{F}_\omega$  a *singular frame* for  $f^\lambda$ . Since  $\hat{B}_\omega$  is normalized, the factors  $\hat{F}_\omega$  and  $\hat{B}_\omega$  depend real analytically on  $z$ , and we can write

$$\hat{B}_\omega = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix} \lambda + o(\lambda^2),$$

where  $\rho$  is a positive real valued function, and  $\mu$  and  $\nu$  are  $\mathbb{C}$ -valued. Now on  $W$ , we have  $\hat{\Phi} = \hat{F}_\omega \hat{B}_\omega \omega_1$ , and since  $\hat{B}_\omega = I$  along  $J$ , we have, for  $z \in J$ ,

$$\begin{aligned} \hat{F}_\omega^{-1} d\hat{F}_\omega &= \hat{\Phi}_\omega^{-1} d\hat{\Phi}_\omega - d\hat{B}_\omega \\ &= \hat{\xi}_\omega - \begin{pmatrix} d\rho & 0 \\ 0 & -\rho^{-2} d\rho \end{pmatrix} - \begin{pmatrix} 0 & d\mu \\ d\nu & 0 \end{pmatrix} \lambda + o(\lambda^2). \end{aligned}$$

Because  $\hat{F}_\omega$  is  $\mathcal{U}$ -valued, it now follows from equation (3.4) and the reality condition defining  $\mathcal{U}$  that, for  $z \in J$ ,

$$\begin{aligned} \hat{F}_\omega^{-1} d\hat{F}_\omega &= \left\{ \begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix} \lambda^{-3} + \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} \lambda^{-2} + \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \lambda^{-1} \right\} dz \\ &\quad + \begin{pmatrix} ir & 0 \\ 0 & -ir \end{pmatrix} dz - \begin{pmatrix} d\rho & 0 \\ 0 & -\rho^{-2} d\rho \end{pmatrix} \\ &\quad + \left\{ \begin{pmatrix} 0 & \bar{b} \\ \bar{a} & 0 \end{pmatrix} \lambda + \begin{pmatrix} \bar{a} & 0 \\ 0 & -\bar{a} \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & -\bar{a} \\ 0 & 0 \end{pmatrix} \lambda^3 \right\} d\bar{z}, \end{aligned}$$

and it is necessary that

$$\begin{pmatrix} 0 & \bar{b} \\ \bar{a} & 0 \end{pmatrix} \lambda dz - \begin{pmatrix} 0 & d\mu \\ d\nu & 0 \end{pmatrix} \lambda = \begin{pmatrix} 0 & \bar{b} \\ \bar{a} & 0 \end{pmatrix} \lambda d\bar{z}.$$

The (1,2) component of this matrix equation is equivalent to

$$\mu_x = 0, \quad \mu_y = 2i\bar{b}.$$

The reality condition for  $\hat{F}_\omega^{-1} d\hat{F}_\omega$  also requires that the (1,1) component of the term constant in  $\lambda$  is pure imaginary, so

$$ir(dx + idy) - \rho_x dx - \rho_y dy = i(pdx + qdy),$$

for some real functions  $p$  and  $q$ . The real part of this equation is equivalent to

$$\rho_x = 0, \quad \rho_y = -r.$$

Writing the  $(1, 1)$  term as  $irdz - (-r)dy = \frac{ir}{2}dz + \frac{ir}{2}d\bar{z}$ , we have just seen that, along  $J$ , the singular frame has Maurer-Cartan form:

$$(3.6) \quad \begin{aligned} \hat{F}_\omega^{-1} d\hat{F}_\omega &= \hat{U}_\omega dz + \hat{V}_\omega d\bar{z}, \\ \hat{U}_\omega &= \begin{pmatrix} -a\lambda^{-2} + \frac{ir}{2} & a\lambda^{-1} \\ -a\lambda^{-3} + b\lambda^{-1} & a\lambda^{-2} - \frac{ir}{2} \end{pmatrix}, \quad \hat{V}_\omega = \begin{pmatrix} \frac{ir}{2} + \bar{a}\lambda^2 & \bar{b}\lambda - \bar{a}\lambda^3 \\ \bar{a}\lambda & -\frac{ir}{2} - \bar{a}\lambda^2 \end{pmatrix}. \end{aligned}$$

Differentiating the Sym-Bobenko formula (2.7), we obtain

$$\begin{aligned} \hat{F}_\omega^{-1} f_z^\lambda \hat{F}_\omega &= -\frac{1}{2H} \left( [\hat{U}_\omega, e_3] + 2i\lambda \frac{\partial}{\partial \lambda} \hat{U}_\omega \right), \\ &= -\frac{2ia\lambda^{-2}}{H} \begin{pmatrix} 1 & -\lambda \\ \lambda^{-1} & -1 \end{pmatrix}, \end{aligned}$$

and similarly,

$$\hat{F}_\omega^{-1} f_{\bar{z}}^\lambda \hat{F}_\omega = \frac{-2i\bar{a}\lambda^2}{H} \begin{pmatrix} 1 & -\lambda \\ \lambda^{-1} & -1 \end{pmatrix}.$$

Adding and subtracting these equations leads to

$$(3.7) \quad \begin{aligned} \hat{F}_\omega^{-1} f_x^\lambda \hat{F}_\omega &= \frac{-4\operatorname{Re}(a\lambda^{-2})}{H} \begin{pmatrix} i & -i\lambda \\ i\lambda^{-1} & -i \end{pmatrix}, \\ \hat{F}_\omega^{-1} f_y^\lambda \hat{F}_\omega &= \frac{4\operatorname{Im}(a\lambda^{-2})}{H} \begin{pmatrix} i & -i\lambda \\ i\lambda^{-1} & -i \end{pmatrix}. \end{aligned}$$

Now, since  $\hat{F}_\omega(z, \bar{z}, \lambda)$  is an element of  $SU_{1,1}$ , it acts by isometries on  $\mathfrak{su}_{1,1} = \mathbb{L}^3$ , and it follows that  $f_x^\lambda$  and  $f_y^\lambda$  are parallel and null. Moreover,  $f_x^\lambda \in \mathbb{L}^3$  is the zero vector if and only if  $\operatorname{Re}(a\lambda^{-2}) = 0$ . Since  $a$  is holomorphic, either the real part of  $a\lambda^{-2}$  is equivalent to zero along the real line, in which case  $f^\lambda(J)$  is a single point, or  $\operatorname{Re}(a\lambda^{-2})$  has isolated zeros on  $J$ , and  $f^\lambda|_J$  is regular away from these zeros.

**Item 5:** By Lemma 2.6,  $\hat{\Phi}$  is in the big cell if and only if

$$(3.8) \quad h := |\mu + \rho|^2 |\rho|^2 - 1 \neq 0.$$

Now we know that for  $z \in J$ , we have  $\rho = 1$  and  $\mu = 0$ , so  $h = |\mu + \rho|^2 |\rho|^2 - 1 = 0$  along  $J$  as expected. To guarantee that  $\Sigma^\circ$  is non-empty, we need to ensure that  $h$  is not constant, and for this it is sufficient to require that  $\frac{\partial h}{\partial y} \neq 0$  at at least one point  $z \in J$ . Using the above expressions for  $\rho_y$  and  $\mu_y$ , and  $\rho = 1$ ,  $\mu = 0$ , one computes

$$\begin{aligned} \frac{\partial h}{\partial y} &= 4\rho_y + (\mu_y + \bar{\mu}_y) \\ &= -4r + 4\operatorname{Im} b. \end{aligned}$$

If this expression is non-zero at  $z_0 \in J$ , then it is also non-zero on a neighbourhood  $\mathcal{N}$  of  $z_0$ , and, because  $h = 0$  and  $h_y \neq 0$  on  $J \cap \mathcal{N}$  it follows that, taking  $\mathcal{N}$  smaller if necessary, the zero set  $C \cap \mathcal{N}$  of  $h|_{\mathcal{N}}$  is precisely  $J \cap \mathcal{N}$ .  $\square$

**Note:** From here on, to simplify notation, we consider mainly  $f = f^1$ , rather than  $f^{\lambda_0}$  for other values of  $\lambda_0 \in \mathbb{S}^1$ . We will also use the convention  $X := \hat{X}|_{\lambda=1}$ , if  $\hat{X}$  depends on  $\lambda$ .

One has the following formulae for the metric and Hopf differential of the surface just constructed:

**Lemma 3.4.** *Let  $f = \widetilde{\mathcal{S}}_1(\hat{\Phi}_\omega) = \widetilde{\mathcal{S}}_1(\hat{\Phi}) : \Sigma_s \rightarrow \mathbb{L}^3$  be a generalized  $H$ -surface constructed from a singular holomorphic frame, factored on  $\hat{\Phi}_\omega^{-1}(\mathcal{B}_{1,1})$  as  $\hat{\Phi}_\omega = \hat{F}_\omega \hat{B}_\omega$  as in Theorem 3.3, and write the Fourier expansion of the matrix valued function  $\hat{B}_\omega \in \widehat{\mathcal{U}}_+^{\mathbb{C}}$  as:*

$$\hat{B}_\omega = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix} \lambda + o(\lambda^2).$$

Let  $\Sigma^\pm := \hat{\Phi}^{-1}(\mathcal{B}_{1,1}^\pm)$ . Then:

(1) *The metric  $ds^2$ , induced by  $f$  on  $\hat{\Phi}_\omega^{-1}(\mathcal{B}_{1,1})$ , is given by the formula*

$$(3.9) \quad ds^2 = 4g^2 (dx^2 + dy^2), \quad g = \varepsilon e^u = \varepsilon \frac{\chi^2 |a|}{H},$$

$$(3.10) \quad \varepsilon(z) = \pm 1, \text{ for } z \in \Sigma^\pm, \quad \chi = \sqrt{|\mu + \rho|^2 - \rho^{-2}}.$$

*The function  $g$  is real analytic on  $\hat{\Phi}_\omega^{-1}(\mathcal{B}_{1,1}) \setminus \mathbb{R}$ , and extends as a  $C^1$  function across the real line. It has the following values at a point  $z_0 \in \mathbb{R} \cap \hat{\Phi}_\omega^{-1}(\mathcal{B}_{1,1})$ :*

$$(3.11) \quad g = 0, \quad \frac{\partial g}{\partial x} = 0, \quad \frac{\partial g}{\partial y} = \frac{4|a|(\text{Im } b - r)}{H}.$$

(2) *The Hopf differential on  $\hat{\Phi}_\omega^{-1}(\mathcal{B}_{1,1})$  is given by  $Qdz$ , where*

$$(3.12) \quad Q = \frac{2a}{H}(b - \tilde{b} - 2ir).$$

*Proof. Item 1:* On  $\hat{\Phi}_\omega^{-1}(\mathcal{B}_{1,1}) \cap \hat{\Phi}^{-1}(\mathcal{B}_{1,1})$  we have, using Lemma 2.6,

$$\begin{aligned} \hat{\Phi} &= \hat{F}_\omega \hat{B}_\omega \omega_1 = \hat{F}_\omega \hat{X} \hat{B}' \\ &= \hat{F} \hat{B}, \end{aligned}$$

where  $\hat{F} = \varepsilon \hat{F}_\omega \hat{X}$ ,  $\hat{B} = \varepsilon \hat{B}'$ , and  $X$  and  $\hat{B}'$  are given in equation (2.16). Writing the Fourier expansion

$$\hat{B} = \begin{pmatrix} \chi & 0 \\ 0 & \chi^{-1} \end{pmatrix} + o(\lambda),$$

the choice of  $u$  and  $\nu$  in  $\hat{B}'$  given in Lemma 2.6 gives the formula (3.10) for  $\chi$ . Since  $\chi > 0$ , this is the unique Iwasawa factorization  $\hat{\Phi} = \hat{F} \hat{B}$  with  $\hat{B} \in \widehat{\mathcal{U}}_+^{\mathbb{C}}$ .

Using this, and the expression (3.5) for  $\hat{\Phi}^{-1} d\hat{\Phi}$ , one obtains

$$\begin{aligned} \hat{F}^{-1} d\hat{F} &= \hat{B} \hat{\Phi}^{-1} d\hat{\Phi} \hat{B}^{-1} + \hat{B} d\hat{B}^{-1} \\ &= \begin{pmatrix} 0 & \chi^2 a \lambda^{-1} \\ \chi^{-2}(b - \tilde{b} - 2ir) \lambda^{-1} & 0 \end{pmatrix} dz + o(\lambda^0). \end{aligned}$$

To calculate the metric, the formulae (2.8), at  $\lambda = 1$ , for  $f_z$  and  $f_{\bar{z}}$  then give:

$$\begin{aligned} f_x &= \frac{2i}{H} F \begin{pmatrix} 0 & \chi^2 a \\ -\chi^2 \bar{a} & 0 \end{pmatrix} F^{-1} \\ &= \frac{2\chi^2 |a|}{H} F_C e_1 F_C^{-1}, \end{aligned}$$

where

$$(3.13) \quad \hat{F}_C := \hat{F} D, \quad D = \begin{pmatrix} e^{i(\frac{\phi}{2} + \frac{\pi}{4})} & 0 \\ 0 & e^{-i(\frac{\phi}{2} + \frac{\pi}{4})} \end{pmatrix}, \quad a = |a| e^{i\phi}.$$

A well-defined choice for the function  $\phi$  can be made because  $a$  is non-vanishing on the simply connected set  $\Sigma$ . Similarly we have

$$f_y = \frac{2\chi^2|a|}{H} F_C e_2 F_C^{-1}.$$

It follows that  $\hat{F}_C$  is the coordinate frame defined by equations (2.2) and that  $2e^u = \frac{2\chi^2|a|}{H}$  (recalling that we have assumed  $H$  is positive), which gives the formula (3.9) for the metric. The factor  $\varepsilon$  is included to achieve continuity of the derivatives of  $g$  across  $\mathbb{R}$ .

The function  $g = \varepsilon \frac{\chi^2|a|}{H}$  is real analytic everywhere on  $\hat{\Phi}_\omega^{-1}(\mathcal{B}_{1,1}) \setminus J$ , because  $\rho$  and  $a$  are non-vanishing and  $g$  is non-vanishing on this set. It has the limiting value zero for  $z \rightarrow J$ , because  $\rho|_J = 1$  and  $\mu|_J = 0$ . To compute the limits of the derivatives at (3.11) for real values of  $z$ , one can differentiate the formula  $\chi = \sqrt{\varepsilon(|\mu + \rho|^2 - \rho^{-2})}$ , with  $\varepsilon = \pm 1$  for  $z \in \Sigma^\pm$ , and use the equations  $\mu_x \rightarrow 0 = \rho_x \rightarrow 0$ ,  $\mu_y \rightarrow 2ib$ ,  $\rho_y \rightarrow -r$ , found in the proof of Theorem 3.3.

**Item 2:** The standard coordinate frame  $\hat{F}_C$ , found above, satisfies

$$\begin{aligned} \hat{F}_C^{-1} d\hat{F}_C &= \begin{pmatrix} 0 & -i\chi^2|a|\lambda^{-1} \\ i\frac{a}{|a|}\chi^{-2}(b - \tilde{b} - 2ir)\lambda^{-1} & 0 \end{pmatrix} dz + o(1), \\ &= \hat{U}dz + \hat{V}d\bar{z}, \end{aligned}$$

where  $\hat{U}$  is given at (2.6). Comparing the off-diagonal components of the above matrix with those of  $\hat{U}$ , and using  $\chi^2 = \frac{e^u H}{|a|}$ , we have

$$i\frac{a}{|a|} \frac{|a|}{He^u} (b - \tilde{b} - 2ir) = \frac{1}{2} ie^{-u} Q,$$

which is the expression (3.12) for  $Q$ . □

**3.2. The converse of Theorem 3.3.** Next we show that every generalized  $H$ -surface that contains a curve in the coordinate domain of its singular set can be locally represented, around that curve, by a standard singular holomorphic potential.

If  $\hat{\Phi} : \Sigma \rightarrow \mathcal{U}^\mathbb{C}$  is a holomorphic map, and  $\hat{\Phi}$  maps at least one point into  $\mathcal{B}_{1,1}$ , then, according to Theorem 2.5, the singular set  $C = \hat{\Phi}^{-1}(\mathcal{P}_1)$  is locally given as the zero set of a non-constant real analytic function  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ . In our setting,  $h$  is given by the formula (3.8),  $h := |\mu + \rho|^2 |\rho|^2 - 1$ .

**Definition 3.5.** A point  $z_0 \in \hat{\Phi}^{-1}(\mathcal{P}_1)$  is said to be a non-degenerate singular point if the derivative map  $dh$  has rank 1 at  $z_0$ , and degenerate if  $dh = 0$ . If, at a point  $z_0 \in \hat{\Phi}^{-1}(\mathcal{P}_1)$  we have the milder condition that there exists a real analytic curve  $\gamma : (-\delta, \delta) \rightarrow \Sigma$ , for some  $\delta > 0$ , with  $\gamma(0) = z_0$  and  $\gamma((-\delta, \delta)) \subset \hat{\Phi}^{-1}(\mathcal{P}_1)$ , then we call  $z_0$  weakly non-degenerate. A generalized  $H$ -surface is non-degenerate or weakly non-degenerate if all singular points have the corresponding property.

For a surface constructed via Theorem 3.3, the non-degeneracy condition is  $\text{Im} b - r \neq 0$ .

**Theorem 3.6.** Let  $f : \Sigma_s \rightarrow \mathbb{L}^3$  be a generalized  $H$ -surface with a corresponding standard potential  $\hat{\xi}$  and holomorphic extended frame  $\hat{\Phi}$ , with  $f = \mathcal{S}_1(\hat{\Phi})$ . Let  $z_0 \in C = \hat{\Phi}^{-1}(\mathcal{P}_1)$  be a weakly non-degenerate singular point. Then, on an open set  $\Omega \subset \Sigma_s$ , containing  $z_0$ , there exist conformal coordinates and a standard singular holomorphic potential  $\hat{\xi}_\omega$ , of the form (3.4), with corresponding singular holomorphic extended frame  $\hat{\Psi}_\omega$ , such that  $f$  is represented on  $\Omega$  by the surface  $\mathcal{S}_1(\hat{\Psi}_\omega)$ .



*Proof.* If  $z_0 \in C$  and  $\hat{\Phi}(z_0) = \hat{F}_0 \omega_1 \hat{B}_0$  is the Iwasawa factorization, set  $\hat{\Phi}_\omega(z) = \hat{\Phi}(z) \hat{B}_0^{-1} \omega_1^{-1}$ . Then  $\hat{\Phi}_\omega(z_0) = \hat{F}_0 \in \mathcal{B}_{1,1}$ , so locally we can Iwasawa factorize  $\hat{\Phi}_\omega(z) = \hat{F}_\omega(z) \hat{B}_\omega(z)$ , with the two factors in  $\mathcal{U}$  and  $\mathcal{W}_+^\mathbb{C}$  respectively. Now

$$(3.14) \quad \hat{\Phi}(z) = \hat{\Phi}_\omega(z) \omega_1 \hat{B}_0 = \hat{F}_\omega(z) \hat{B}_\omega(z) \omega_1 \hat{B}_0,$$

and this is in the big cell precisely when  $\hat{B}_\omega(z) \omega_1$  is. As  $z_0$  is weakly non-degenerate, there is a curve through  $z_0$  which is mapped by  $\hat{\Phi}$  into  $\mathcal{P}_1$ . After a conformal change of coordinates (taking a smaller neighbourhood if necessary) we can assume that this curve is an open interval  $J$  on the line  $\{y = 0\} \subset \mathbb{C}$ , and that  $z_0$  is the origin. By Lemma 2.6, we can, on the interval  $J$ , write

$$\hat{B}_\omega(x, 0) \omega_1 = R_\theta(x) \omega_1 \tilde{B}(x), \quad R_\theta(x) := \begin{pmatrix} e^{-i\theta(x)/2} & 0 \\ 0 & e^{i\theta(x)/2} \end{pmatrix} \in \mathcal{U}, \quad \tilde{B}(x) \in \mathcal{W}_+^\mathbb{C},$$

where  $R_\theta$  and  $\tilde{B}$  are real analytic in  $x$ . Substituting into equation (3.14), this means

$$\hat{\Phi}|_J(x) = F_*(x) \omega_1 B_*(x), \quad F_*(x) := \hat{F}_\omega(x, 0) R_\theta(x), \quad B_*(x) := \tilde{B}(x) \hat{B}_0.$$

Now, by extending  $\theta(x)$  analytically,  $R_\theta$  has a holomorphic extension  $\check{R}_\theta : \Omega \rightarrow \mathcal{U}^\mathbb{C}$  to some open set  $\Omega$  containing  $I$ . Similarly, since the Maurer-Cartan form of  $\hat{F}_\omega|_J$ , has only a finite number of real analytic functions in its Fourier expansion in  $\lambda$ , this map also has a holomorphic extension to a map  $\check{F}_\omega : \Omega \rightarrow \mathcal{W}^\mathbb{C}$ , taking  $\Omega$  sufficiently small. Therefore  $B_* = \omega_1^{-1} \cdot R_\theta^{-1} \cdot \hat{F}_\omega^{-1}|_J \cdot \hat{\Phi}|_J$  extends holomorphically to a map  $B_* : \Omega \rightarrow \mathcal{W}_+^\mathbb{C}$ , given by  $B_*(z) = \omega_1^{-1} \cdot \check{R}_\theta^{-1}(z) \cdot \check{F}_\omega^{-1}(z) \cdot \hat{\Phi}(z)$ . This allows one to define a holomorphic map

$$\begin{aligned} \hat{\Psi}(z) &:= \hat{\Phi}(z) B_*^{-1}(z) \\ &= \check{F}_\omega(z) \check{R}_\theta(z) \omega_1. \end{aligned}$$

This has the property that  $\widetilde{\mathcal{S}}(\hat{\Psi}(z)) = \widetilde{\mathcal{S}}(\hat{\Phi}(z))$ , because  $B_*^{-1}(z) \in \mathcal{W}_+^\mathbb{C}$  and therefore has no impact on the Iwasawa decomposition of  $\hat{\Phi}$ . Moreover, it is easy to verify that  $\hat{\Psi}^{-1} d\hat{\Psi}$  is also a *standard* holomorphic potential, because right multiplication by a holomorphic map into  $\mathcal{W}_+^\mathbb{C}$  preserves the relevant properties. Finally, consider the translate,  $\hat{\Psi}_\omega := \hat{\Psi} \omega_1^{-1}$ . By definition, we have

$$\hat{\Psi}_\omega|_J(x) = F_*(x) \in \mathcal{U}.$$

Hence, as shown in Section 3.1, it follows that  $\hat{\xi}_\omega := \hat{\Psi}_\omega^{-1} d\hat{\Psi}_\omega$  is a singular holomorphic potential of the form given by (3.4). By construction, we have, on the open set  $\Omega$ ,

$$\widetilde{\mathcal{S}}_1(\hat{\Psi}_\omega) = \widetilde{\mathcal{S}}_1(\hat{\Psi}) = \widetilde{\mathcal{S}}_1(\hat{\Phi}) = f.$$

□

#### 4. PRESCRIBING SINGULARITIES: THE SINGULAR BJÖRLING PROBLEM

We showed that if  $f : \Sigma_s \rightarrow \mathbb{L}^3$  is a generalized  $H$ -surface, and  $z_0 \in \Sigma_s$  is a weakly non-degenerate singular point, then, at least locally,  $f$  can be constructed from a singular frame  $\hat{F}_\omega$  which satisfies the equations (3.7), which, at  $\lambda = 1$ , are:

$$(4.1) \quad F_\omega^{-1} f_x F_\omega = \frac{-4\operatorname{Re}(a)}{H} (-e_2 + e_3), \quad F_\omega^{-1} f_y F_\omega = \frac{4\operatorname{Im}(a)}{H} (-e_2 + e_3).$$

The singular Björling problem can be stated as the task of constructing the singular frame  $\hat{F}_\omega$  – and hence the surface – given that we only know  $f$  (and therefore  $f_x$ , if  $x$  is the parameter of the curve) and  $f_y$  along the singular curve.

So suppose we have an open set  $\Omega \subset \mathbb{C}$ , with coordinates  $z = x + iy$ , and such that  $J = \Omega \cap \mathbb{R} = (x_1, x_2)$  is a non-empty open interval containing the origin. Suppose there exists a generalized  $H$ -surface  $f : \Omega \rightarrow \mathbb{L}^3$ , satisfying the Björling data along  $J$ , and with associated holomorphic extended frame  $\hat{\Phi}$ , such that  $\hat{\Phi}(J) \subset \mathcal{P}_1$ . Since the vector fields  $f_x$  and  $f_y$  are both necessarily null and parallel along  $J$ , we can, on this interval, and after an isometry of  $\mathbb{L}^3$ , write

$$f_x = s \begin{pmatrix} i & e^{i\theta} \\ e^{-i\theta} & -i \end{pmatrix}, \quad f_y = t \begin{pmatrix} i & e^{i\theta} \\ e^{-i\theta} & -i \end{pmatrix}, \quad \theta(0) = -\frac{\pi}{2},$$

where  $s$ ,  $\theta$  and  $t$  are all real analytic functions  $J \rightarrow \mathbb{R}$ . We assume that  $s$  and  $t$  never vanish at the same time, so that  $\theta$  is well defined on  $J$ .

The equations (4.1) suggest that we choose a frame  $F_0$  to be the rotation about the  $x_3$ -axis which rotates  $[\cos \theta, \sin \theta, 0]^T \in \mathbb{L}^3$  so that it points in the  $-e_2$  direction:

$$(4.2) \quad F_0 = \begin{pmatrix} e^{i\frac{2\theta+\pi}{4}} & 0 \\ 0 & e^{-i\frac{2\theta+\pi}{4}} \end{pmatrix}.$$

The normalization of  $\theta$  means that  $F_0(0) = I$ . Then

$$(4.3) \quad F_0^{-1} f_x F_0 = s(-e_2 + e_3), \quad F_0^{-1} f_y F_0 = t(-e_2 + e_3).$$

Comparing this with equations (4.1), we must have, along  $J$ ,

$$\operatorname{Re} a = -\frac{Hs}{4}, \quad \operatorname{Im} a = \frac{Ht}{4}.$$

Thus our regularity assumption on  $s$  and  $t$  is actually equivalent to the assumption that the surface is a generalized  $H$ -surface, i.e.  $a$  is non-vanishing.

To find the  $\lambda$  dependence of the singular frame, we know from equation (3.3) that this frame satisfies:

$$(4.4) \quad \hat{F}_\omega^{-1} d\hat{F}_\omega = \left\{ \begin{pmatrix} -a\lambda^{-2} & a\lambda^{-1} \\ -a\lambda^{-3} + b\lambda^{-1} & a\lambda^{-2} \end{pmatrix} + \begin{pmatrix} ir & 0 \\ 0 & -ir \end{pmatrix} + \begin{pmatrix} \bar{a}\lambda^2 & \bar{b}\lambda - \bar{a}\lambda^3 \\ \bar{a}\lambda & -\bar{a}\lambda^2 \end{pmatrix} \right\} dx.$$

Evaluating at  $\lambda = 1$  and comparing this with the Maurer-Cartan form of our frame:

$$F_0^{-1} dF_0 = \begin{pmatrix} \frac{i}{2}\theta_x & 0 \\ 0 & -\frac{i}{2}\theta_x \end{pmatrix} dx,$$

and using the above formula for  $\operatorname{Im} a$ , we obtain along  $J$  the values :  $r = \frac{1}{2}(\theta_x + Ht)$ , and  $b = \frac{1}{2}iHt$ . Substituting  $a$ ,  $b$  and  $r$  into equation (4.4) and extending holomorphically, gives the singular holomorphic potential  $\hat{\xi}_\omega$ . The non-degeneracy condition  $r - \operatorname{Im} b \neq 0$  for the singular curve is

$$(4.5) \quad \theta_x \neq 0.$$

**Theorem 4.1.** *Suppose given a real analytic function  $f_0 : J \rightarrow \mathbb{L}^3$ , such that  $\frac{df_0}{dx}$  is a null vector field, and an additional null real analytic vector field  $v(x)$ , such that  $v(x)$  is a scalar multiple of  $\frac{df_0}{dx}(x)$  for each  $x \in J$ . Suppose also that the vector fields do not vanish simultaneously at any point  $x \in J$ . Let  $s$  and  $t$  be defined as above. Let  $\hat{\Phi}_\omega$  be the singular holomorphic frame obtained by analytically extending the 1-form  $\hat{F}_\omega^{-1} d\hat{F}_\omega$  given by (4.4), with*

$$a = \frac{H}{4}(-s + it), \quad b = \frac{1}{2}iHt, \quad r = \frac{1}{2}(\theta_x + Ht),$$

to some simply connected open set containing  $J$ , and integrating with initial condition  $\hat{\Phi}_\omega(0) = I$ . Suppose that  $\hat{\Phi} = \hat{\Phi}_\omega \omega_1$  maps at least one point into  $\mathcal{B}_{1,1}$ . Then the surface

$$f(x, y) := \widetilde{\mathcal{S}}_1(\hat{\Phi}_\omega(x, y)) + \frac{1}{2H} e_3 + f_0(0),$$

is the unique weakly non-degenerate generalized  $H$ -surface such that  $f$ ,  $f_x$  and  $f_y$  coincide respectively with  $f_0$ ,  $\frac{df_0}{dx}$  and  $v$  along the real interval  $J$ .

Uniqueness here is understood to mean that the two surfaces are both defined and agree on some open subset of  $\mathbb{C}$  containing the interval  $J$ . We remark that a condition that guarantees that  $\hat{\Phi}$  maps at least one point into the big cell is that  $\frac{d^2 f_0}{dx^2}$  is not parallel to  $\frac{df_0}{dx}$  (that is,  $\theta_x \neq 0$ ) at some point on  $J$ .

*Proof.* By construction, and with the assumption that  $\hat{\Phi}^{-1}(\mathcal{B}_{1,1})$  is non-empty,  $f$  is a generalized  $H$ -surface that has the required values along  $J$ , so we need to show uniqueness.

Suppose  $\tilde{f}$  is another generalized  $H$ -surface satisfying the Björling data. It is necessarily weakly non-degenerate. By Theorem 3.6, there exists a standard singular holomorphic potential  $\tilde{\xi}_\omega$  and singular holomorphic frame  $\hat{\Psi}_\omega$  such that  $\widetilde{\mathcal{S}}_1(\hat{\Psi}_\omega) = \tilde{f} + \text{translation}$ . No coordinate change is necessary, since the condition that  $\tilde{f}$  is not immersed along  $J$  implies that the holomorphic extended frame defining  $\tilde{f}$  already maps  $J$  into  $\mathcal{P}_1$ .

Let  $\hat{G}_\omega$  be the singular frame obtained by the Iwasawa decomposition  $\hat{\Psi}_\omega = \hat{G}_\omega \hat{B}_\omega$ , with  $\hat{B}_\omega \in \widehat{\mathcal{U}}_+^{\mathbb{C}}$ . As shown in the proof of Theorem 3.3, the map  $\tilde{f}$  satisfies, at points  $z \in J$ ,

$$\hat{G}_\omega^{-1} \tilde{f}_x \hat{G}_\omega = \frac{-4\operatorname{Re}(A\lambda^{-2})}{H} \begin{pmatrix} i & -i\lambda \\ i\lambda^{-1} & -i \end{pmatrix}, \quad \hat{G}_\omega^{-1} \tilde{f}_y \hat{G}_\omega = \frac{4\operatorname{Im}(A\lambda^{-2})}{H} \begin{pmatrix} i & -i\lambda \\ i\lambda^{-1} & -i \end{pmatrix},$$

where  $\hat{\Psi}^{-1} d\hat{\Psi} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \lambda^{-1} dz + o(\lambda)$ , and  $\hat{\Psi} := \hat{\Psi}_\omega \omega_1$ . On the other hand, we have, by assumption that  $\tilde{f}_x$  and  $\tilde{f}_y$  satisfy the equations (4.3), namely, along  $J$ ,

$$\tilde{f}_x = sF_0(-e_2 + e_3)F_0^{-1}, \quad \tilde{f}_y = tF_0(-e_2 + e_3)F_0^{-1}.$$

We will first show that we can assume, without loss of generality, that  $\operatorname{Re} A = -\frac{Hs}{4}$  and  $\operatorname{Im} A = \frac{Ht}{4}$  as follows: comparing the equations above, it follows that, wherever  $s \neq 0 \neq t$  we have

$$\frac{t}{\operatorname{Im} A} = \frac{-s}{\operatorname{Re} A} =: \kappa.$$

At least one of  $s(x)$  or  $t(x)$  is non-zero at each  $x \in J$ , and so  $\kappa : J \rightarrow \mathbb{R}$  is well defined and non-vanishing. Let  $\beta$  be the holomorphic extension of  $\frac{\sqrt{\kappa H}}{2}$  to a simply connected open set  $\mathcal{N} \subset \mathbb{C}$  which contains  $J$ . Set

$$\hat{\Psi}' := \hat{\Psi} \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta \end{pmatrix}.$$

Then  $\widetilde{\mathcal{S}}(\hat{\Psi}') = \widetilde{\mathcal{S}}(\hat{\Psi})$  because the  $\mathcal{U}$  factor in the Iwasawa factorization is the same for both of these. So we can replace  $\hat{\Psi}$  by  $\hat{\Psi}'$  and we have

$$(\hat{\Psi}')^{-1} d\hat{\Psi}' = \begin{pmatrix} 0 & a \\ \beta^{-2} B & 0 \end{pmatrix} \lambda^{-1} dz + o(\lambda),$$

where  $a = \frac{H}{4}(-s + it)$  on  $J$ . The new singular frame  $\hat{G}'_\omega$ , which is obtained from the factorization of  $\hat{\Psi}' \omega_1^{-1} =: \hat{\Psi}'_\omega = \hat{G}'_\omega \hat{B}'_\omega$  satisfies  $\mathcal{S}_1(\hat{G}'_\omega) = \mathcal{S}_1(\hat{G}_\omega) = \tilde{f} + \text{translation}$ , but the frame

now also satisfies, along  $J$ , the analogue of equations (4.6), replacing  $A$  with  $a = \frac{H}{4}(-s + it)$ . But the frame  $\hat{F}_\omega$  constructed above for  $f$  also satisfies the same equations. This implies that

$$\hat{F}_\omega^{-1} \hat{G}'_\omega|_J = \hat{T},$$

where  $\hat{T} : J \rightarrow \mathcal{U}_1$  commutes with the matrix  $\begin{pmatrix} i & -i\lambda \\ i\lambda^{-1} & -i \end{pmatrix}$ . A computation (using that all matrices are normalized to  $I$  at  $z = 0$ ), shows that  $\hat{T}$  must be of the form

$$\hat{T} = \begin{pmatrix} 1 - iR & iR \\ -iR & 1 + iR \end{pmatrix}, \quad R : J \times \mathbb{S}^1 \rightarrow \mathbb{R},$$

where  $R$  depends on the loop parameter  $\lambda$ . Now

$$\begin{aligned} \mathcal{S}_1(\hat{G}'_\omega) &= \mathcal{S}_1(\hat{F}_\omega \hat{T}) \\ &= \frac{-1}{2H} \left( \hat{F}_\omega \hat{T} e_3 \hat{T}^{-1} \hat{F}_\omega^{-1} + 2i\lambda \left( \frac{\partial}{\partial \lambda} \hat{F}_\omega \right) \hat{F}_\omega^{-1} + 2i\lambda \hat{F}_\omega \left( \frac{\partial}{\partial \lambda} \hat{T} \right) \hat{T}^{-1} \hat{F}_\omega^{-1} \right) \Big|_{\lambda=1} \\ &= \mathcal{S}_1(\hat{F}_\omega) - \frac{1}{H} \left( \hat{F}_\omega \begin{pmatrix} iR^2 & R - iR^2 \\ R + iR^2 & -iR^2 \end{pmatrix} \hat{F}_\omega^{-1} + i\lambda \hat{F}_\omega \frac{\partial R}{\partial \lambda} \begin{pmatrix} -i & i \\ -i & i \end{pmatrix} \hat{F}_\omega^{-1} \right) \Big|_{\lambda=1}. \end{aligned}$$

We can use the assumption that  $\tilde{f} = f$  along  $J$ , that is,  $\mathcal{S}_1(\hat{G}'_\omega) = \mathcal{S}_1(\hat{F}_\omega) + \text{translation, along } J$ . Since all maps are normalized to the identity at  $z = 0$ , this translation is actually the zero vector. It follows from this and the formula for  $\mathcal{S}_1(\hat{G}'_\omega)$  that

$$\left( \begin{pmatrix} iR^2 & R - iR^2 \\ R + iR^2 & -iR^2 \end{pmatrix} + i\lambda \frac{\partial R}{\partial \lambda} \begin{pmatrix} -i & i \\ -i & i \end{pmatrix} \right) \Big|_{\lambda=1} = 0.$$

This gives the pair of equations

$$\left( iR^2 + \frac{\partial R}{\partial \lambda} \right) \Big|_{\lambda=1} = 0, \quad \left( R - iR^2 - \frac{\partial R}{\partial \lambda} \right) \Big|_{\lambda=1} = 0.$$

Hence  $R|_{\lambda=1} = 0$ , that is,

$$G'_\omega|_J = F_\omega|_J = F_0.$$

But we already saw, in the paragraphs preceding this theorem, that, given that we know the value of  $a$  along  $J$ , the singular frame  $\hat{F}_\omega$  is then uniquely determined by its value  $F_0$  along  $J$ . Hence  $\hat{G}'_\omega = \hat{F}_\omega$ , and  $\tilde{f} = f$ .  $\square$

**4.1. Example.** Choose  $I = \mathbb{R}$ , and the singular curve to be the helix in  $\mathbb{L}^3$  given by  $f_0(x) = [\sin(x), -\cos(x), x]^T$ ,  $f_x = [\cos(x), \sin(x), 1]$  and  $v(x) = f_x(x)$ . Then  $\theta(x) = x$ ,  $s = t = 1$  along  $\mathbb{R}$ . We have  $a = \frac{H}{4}(-1 + i)$ ,  $b = \frac{1}{2}iH$  and  $r = \frac{1}{2}(1 + H)$ . The singular potential is

$$\begin{aligned} \hat{\xi}_\omega &= \frac{H}{4} \left\{ \begin{pmatrix} (1-i)\lambda^{-2} & (-1+i)\lambda^{-1} \\ (1-i)\lambda^{-3} + 2i\lambda^{-1} & -(1-i)\lambda^{-2} \end{pmatrix} + \begin{pmatrix} 2i(1 + \frac{1}{H}) & 0 \\ 0 & -2i(1 + \frac{1}{H}) \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} -(1+i)\lambda^2 & -2i\lambda + (1+i)\lambda^3 \\ -(1+i)\lambda & (1+i)\lambda^2 \end{pmatrix} \right\} dz. \end{aligned}$$

The corresponding translated frame,  $\hat{\Phi} = \hat{\Phi}_\omega \omega_1$  has, from equation (3.5), standard potential:

$$\hat{\Phi}^{-1} d\hat{\Phi} = \frac{H}{4} \begin{pmatrix} \frac{2i}{H} & (-1+i)\lambda^{-1} - 2i\lambda + (1+i)\lambda^3 \\ -4i(1 + \frac{1}{H})\lambda^{-1} & -\frac{2i}{H} \end{pmatrix} dz.$$

## 5. IDENTIFYING SINGULARITY TYPES VIA THE BJÖRLING CONSTRUCTION

In this section we find the conditions on the Björling data for the surface constructed to have a cuspidal edge, swallowtail or cuspidal cross cap singularity in a neighbourhood of a singular point. If one considers non-degenerate  $H$ -surfaces parameterized by germs of their Björling data at some point, then one can see that these are the generic singularities within this class. However, see the comments in Section 1.5.

We first show that every weakly non-degenerate  $H$ -surface is a frontal, and then use the criteria in [16] and [11] for a frontal to have these types of singularities. Examples are illustrated in Figure 2.

**5.1. The Euclidean normal to a generalized  $H$ -surface.** The commutators of our basis matrices satisfy  $[e_1, e_2] = -2e_3$ ,  $[e_2, e_3] = 2e_1$ , and  $[e_3, e_1] = 2e_2$ , and from this it follows that the Euclidean cross-product on the vector space  $\mathbb{R}^3$  corresponding to  $\mathbb{L}^3$  is given by

$$A \times B = -\frac{1}{2} \text{Ad}_{e_3}[A, B],$$

where  $[\cdot, \cdot]$  is the matrix commutator, and  $\text{Ad}_X$  denotes conjugation by  $X$ . Let  $\|\cdot\|_E$  denote the standard Euclidean norm on  $\mathbb{R}^3$ .

Let  $f$  be a generalized  $H$ -surface with holomorphic frame  $\hat{\Phi}$ . Since  $f_x$  and  $f_y$  are parallel at singular points, the cross-product of these vanishes there. Recall that the big cell is the union of two disjoint open sets,  $\mathcal{B}_{1,1} = \mathcal{B}_{1,1}^+ \cup \mathcal{B}_{1,1}^-$ . It turns out that one achieves continuity across the singular set  $C$  by defining, on  $\Sigma^\circ$ , the *Euclidean (unit) normal* as follows:

$$\mathbf{n}_E(z) := \varepsilon \frac{f_x \times f_y}{\|f_x \times f_y\|_E}(z), \quad \varepsilon(z) = \pm 1, \text{ for } z \in \hat{\Phi}^{-1}(\mathcal{B}_{1,1}^\pm).$$

The two sets  $\hat{\Phi}^{-1}(\mathcal{B}_{1,1}^\pm)$  are open and disjoint, so  $\mathbf{n}_E$  is a real analytic vector field on  $\Sigma^\circ$ .

**Lemma 5.1.** *Let  $f : \Sigma_s \rightarrow \mathbb{L}^3$  be a weakly non-degenerate generalized  $H$ -surface. Then the Euclidean unit normal extends across  $C = \hat{\Phi}^{-1}(\mathcal{P}_1)$  to give a real analytic vector field on  $\Sigma_s$ . At a point  $z_0 \in C$ , if coordinates are chosen so that the singular holomorphic frame  $\hat{\Phi}_\omega$  defined in Theorem 3.6 satisfies  $\hat{\Phi}_\omega(z_0) = I$ , then the Euclidean normal is given at  $z_0$  by*

$$(5.1) \quad \mathbf{n}_E(z_0) = \frac{1}{\sqrt{2}}(e_2 + e_3).$$

*If  $\hat{F}_\omega$  is the singular frame obtained from  $\hat{\Phi}_\omega$  then, at nearby singular values  $z \in C$ , the Euclidean normal is the unit vector in the direction of*

$$(5.2) \quad \widetilde{\mathbf{n}}_E = \text{Ad}_{e_3} F_\omega(-e_2 + e_3) F_\omega^{-1}.$$

*Proof.* On a neighbourhood,  $\Omega \subset \Sigma$ , of  $z_0 \in C$  we can assume by Theorem 3.6 that  $f$  is defined by a standard singular holomorphic frame  $\hat{\Phi}_\omega$  with  $\hat{\Phi}_\omega(z_0) = I$ , with coordinates such that  $z_0 = 0$ , and that there is an interval  $J = \Omega \cap \mathbb{R}$  containing 0 such that  $J \subset C$ . On an open dense subset,  $\Omega^\circ = \Omega \cap \Sigma^\circ$ , of  $\Omega$ , we can Iwasawa factorize the standard holomorphic frame  $\hat{\Phi} = \hat{\Phi}_\omega \omega_1$  as  $\hat{\Phi} = \hat{F} \hat{B}$ , with  $\hat{F} \in \mathcal{U}$ ,  $\hat{B} \in \widehat{\mathcal{U}}_+^{\mathbb{C}}$ . Now we have,

$$\begin{aligned} f_x &= |f_x| F_C e_1 F_C^{-1}, & f_y &= |f_y| F_C e_2 F_C^{-1} \\ &= |f_x| F D e_1 D^{-1} F^{-1}, & f_y &= |f_y| F D e_2 D^{-1} F^{-1} \end{aligned}$$

where  $F_C$  and  $D$  are given at equation (3.13), and so  $\mathbf{n}_E$  points in the direction of

$$\begin{aligned} \widetilde{X} &= \varepsilon (F D e_1 D^{-1} F^{-1}) \times (F D e_2 D^{-1} F^{-1}) \\ &= \varepsilon \text{Ad}_{e_3}(F e_3 F^{-1}). \end{aligned}$$

As in the proof of Lemma 3.4, by Lemma 2.6, we have

$$\begin{aligned}\hat{\Phi} &= \hat{F}\hat{B} = \hat{\Phi}_\omega\omega_1 = \hat{F}_\omega\hat{B}_\omega\omega_1 \\ &= \hat{F}_\omega\hat{K}\hat{B}',\end{aligned}$$

where  $\hat{F} = \varepsilon\hat{F}_\omega\hat{K}$ ,  $\hat{B}_\omega = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix}\lambda + o(\lambda^2)$ , and  $\rho : \Omega \rightarrow \mathbb{R}_+$ , and  $\mu$  and  $\nu$  are  $\mathbb{C}$ -valued. We also have

$$\hat{B}_\omega|_J = I, \quad \hat{F}_\omega(0) = I.$$

On  $\Omega^\circ$  we can write

$$\tilde{X} = \varepsilon \text{Ad}_{e_3}(F_\omega K e_3 K^{-1} F_\omega^{-1}).$$

Since  $F_\omega$  is real analytic on the whole of  $\Omega$ , we only need to analyze  $\tilde{Y} := \varepsilon K e_3 K^{-1}$ . According to Lemma 2.6, we can choose  $\hat{K}$  as

$$\begin{aligned}\hat{K} &= \begin{pmatrix} u & \nu\lambda \\ \varepsilon\bar{\nu}\lambda^{-1} & \varepsilon\bar{u} \end{pmatrix}, \\ \nu &= \frac{1}{\sqrt{\varepsilon h}}, \quad u = \varepsilon(\mu + \rho)\rho\bar{\nu}, \\ h &:= |\mu + \rho|^2|\rho|^2 - 1, \quad |u|^2 - |\nu|^2 = \varepsilon.\end{aligned}$$

Then

$$\tilde{Y} = \varepsilon \begin{pmatrix} i\varepsilon(u\bar{u} + \nu\bar{\nu}) & -2iuv \\ 2i\bar{u}\bar{\nu} & -i\varepsilon(u\bar{u} + \nu\bar{\nu}) \end{pmatrix},$$

and

$$\begin{aligned}\|\tilde{Y}\|_E^2 &= (|u|^2 + |\nu|^2)^2 + 4|u|^2|\nu|^2 \\ &= (\varepsilon + 2|\nu|^2)^2 + 4(\varepsilon + |\nu|^2)|\nu|^2 \\ &= 1 + \frac{8}{h} \left(1 + \frac{1}{h}\right).\end{aligned}$$

The unit vector in the direction of  $\tilde{Y}$  is

$$\begin{aligned}Y &= (1 + 8h^{-1}(1 + h^{-1}))^{-\frac{1}{2}}\tilde{Y} \\ (5.3) \quad &= i \begin{pmatrix} Z_1 + Z_2 & -(\mu + \rho)\rho Z_1 \\ (\bar{\mu} + \rho)\rho Z_1 & -Z_1 - Z_2 \end{pmatrix},\end{aligned}$$

where

$$(5.4) \quad Z_1 := \varepsilon 2h^{-1}(1 + 8h^{-1}(1 + h^{-1}))^{-\frac{1}{2}}, \quad \lim_{h \rightarrow 0} Z_1 = \frac{1}{\sqrt{2}}, \quad \lim_{h \rightarrow 0} \frac{\partial Z_1}{\partial y} = -\frac{1}{2\sqrt{2}}h_y,$$

$$(5.5) \quad Z_2 := \varepsilon(1 + 8h^{-1}(1 + h^{-1}))^{-\frac{1}{2}}, \quad \lim_{h \rightarrow 0} Z_2 = 0, \quad \lim_{h \rightarrow 0} \frac{\partial Z_2}{\partial y} = \frac{1}{2\sqrt{2}}h_y.$$

Thus  $Y$  is a well-defined real analytic vector field which, for real values of  $z$ , that is when  $h = \mu = 0$  and  $\rho = 1$ , has the value

$$Y(x) = \frac{1}{\sqrt{2}}(-e_2 + e_3).$$

Substituting this for  $\varepsilon K e_3 K^{-1}$  in the expression for  $\tilde{X}$  above, gives the stated formulae for  $\mathbf{n}_E(z_0)$  and  $\widetilde{\mathbf{n}}_E(x)$ .  $\square$

**Lemma 5.2.** *Let  $f$  be a generalized  $H$ -surface constructed from the Björling data in Theorem 4.1. At  $z = 0$ , the derivative  $\mathbf{dn}_E$  of the Euclidean unit normal is given by*

$$(5.6) \quad \mathbf{dn}_E = -\frac{\theta_x}{\sqrt{2}}e_1 dx - \frac{Ht}{2\sqrt{2}}(-e_2 + e_3)dy.$$

*Proof.* We showed in the previous lemma that  $\mathbf{n}_E = \beta X$ , for some real-valued function  $\beta$  and  $X = \text{Ad}_{e_3}(F_\omega Y F_\omega^{-1})$ , where  $Y$  is given by equation (5.3). We also have that  $X(0) = \mathbf{n}_E(0)$ , which means that  $\beta(0) = 1$ . Now  $\langle \mathbf{n}_E, \mathbf{dn}_E \rangle_E = 0$ , and  $X$  is parallel to  $\mathbf{n}_E$ , so it follows that

$$\begin{aligned} \mathbf{dn}_E &= d\beta X + \beta dX \\ &= \beta (dX - \langle dX, \mathbf{n}_E \rangle_E \mathbf{n}_E), \end{aligned}$$

and we need to compute

$$dX = \text{Ad}_{e_3}(F_\omega [F_\omega^{-1} dF_\omega, Y] F_\omega^{-1}) + \text{Ad}_{e_3}(F_\omega dY F_\omega^{-1}).$$

At  $z = 0$ , we have, using  $U_\omega$  and  $V_\omega$  from (3.6),

$$F_\omega^{-1}(F_\omega)_x = \frac{i\theta_x}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F_\omega^{-1}(F_\omega)_y = i(U_\omega - V_\omega) = \begin{pmatrix} \frac{Hs}{2}i & -\frac{Ht}{2} - \frac{Hs}{2}i \\ -\frac{Ht}{2} + \frac{Hs}{2}i & -\frac{Hs}{2}i \end{pmatrix},$$

and, by the formulae  $h_y = -4(r - \text{Im} b)$ ,  $\mu_y = 2i\bar{b}$  and  $\rho_y = -r$  from the proof of Theorem 3.3,

$$\begin{aligned} h &= 0, & h_x &= 0, & h_y &= -2\theta_x, \\ \mu &= 0, & \mu_x &= 0, & \mu_y &= Ht, \\ \rho &= 1, & \rho_x &= 0, & \rho_y &= -\frac{1}{2}(\theta_x + Ht). \end{aligned}$$

Using these and the formulae (5.3)-(5.5) one obtains, at  $z = 0$ ,

$$X_x = -\frac{\theta_x}{\sqrt{2}}e_1, \quad X_y = -\frac{Ht}{\sqrt{2}}e_3.$$

Together with the value  $\mathbf{n}_E = \frac{1}{\sqrt{2}}(e_2 + e_3)$  at  $z = 0$ , and  $\beta(0) = 1$ , this gives the expression (5.6) for  $\beta (dX - \langle dX, \mathbf{n}_E \rangle_E \mathbf{n}_E)|_{z=0}$ .  $\square$

**Lemma 5.3.** *Let  $f : \Sigma_s \rightarrow \mathbb{L}^3$  be a generalized  $H$ -surface constructed by the data in Theorem 3.3, with  $\hat{\Phi}_\omega(0) = \hat{\Phi}(0)\omega_1^{-1} = I$ . Set  $s_0 := -\frac{4\text{Re}a(0)}{H}$  and  $t_0 := \frac{4\text{Im}a(0)}{H}$  so that*

$$f_x = s_0(-e_2 + e_3), \quad f_y = t_0(-e_2 + e_3),$$

*Let  $\psi : \Sigma_s \rightarrow \mathbb{R}$  be defined by*

$$\psi = \varepsilon \|f_x \times f_y\|_E,$$

*where  $\varepsilon(z) = \pm 1$ , for  $z \in \Phi^{-1}(\mathcal{B}_{1,1}^\pm)$ . Then at  $z = 0$ ,*

$$(5.7) \quad d\psi = \frac{16|a|(\text{Im} b - r)}{H} \sqrt{\frac{t_0^2 + s_0^2}{2}} dy.$$

*In particular,  $d\psi(0) = 0 \Leftrightarrow \text{Im} b(0) - r(0) = 0$ .*

*Proof.* At points away from the real line, we have the decomposition  $\hat{\Phi} = \hat{F}\hat{B}$ , and the coordinate frame found in Lemma 3.4 is:  $\hat{F}_C := \hat{F}D$ , with  $D = \text{diag}\left(e^{i(\frac{\theta}{2} + \frac{\pi}{4})}, e^{-i(\frac{\theta}{2} + \frac{\pi}{4})}\right)$ , and  $a = |a|e^{i\phi}$ . The metric is given by  $ds^2 = 4g^2(dx^2 + dy^2)$ , with  $g = \varepsilon \frac{\chi^2|a|}{H}$  and  $\chi = \sqrt{|\mu + \rho|^2 - \rho^{-2}}$ . And we have:

$$(5.8) \quad f_x = 2\varepsilon g F_C e_1 F_C^{-1}, \quad f_y = 2\varepsilon g F_C e_2 F_C^{-1}, \quad N = F_C e_3 F_C^{-1},$$

where  $N$  is the Lorentzian unit normal. Now

$$\begin{aligned} f_x \times f_y &= -\frac{1}{2} \text{Ad}_{e_3} [f_x, f_y] \\ &= 4g^2 \text{Ad}_{e_3} F_C e_3 F_C^{-1}, \end{aligned}$$

so we can write  $\psi = \varepsilon \|f_x \times f_y\|_E$  as

$$\psi = 4g\Gamma, \quad \Gamma := \varepsilon g \|N\|_E.$$

Although  $g \rightarrow 0$  and  $\|N\|_E \rightarrow \infty$  as  $z \rightarrow \mathbb{R}$ , we can get an explicit expression for the product  $\Gamma$ .

Writing  $F_C = \begin{pmatrix} A & B \\ \varepsilon \bar{B} & \varepsilon \bar{A} \end{pmatrix}$ , the equations (5.8) then imply that, as  $z \rightarrow 0$ , we have the finite limits:

$$\begin{aligned} g \operatorname{Im}(A\bar{B}) &\rightarrow -\frac{s_0}{4}, & \varepsilon g (A^2 - B^2) &\rightarrow -i\frac{s_0}{2}, \\ g \operatorname{Re}(A\bar{B}) &\rightarrow -\frac{t_0}{4}, & \varepsilon g (A^2 + B^2) &\rightarrow -\frac{t_0}{2}, \end{aligned}$$

which imply

$$\varepsilon g A^2 \rightarrow -\frac{1}{4}(t_0 + is_0), \quad \varepsilon g B^2 \rightarrow \frac{1}{4}(-t_0 + is_0).$$

Now

$$N = i \begin{pmatrix} \varepsilon(|A|^2 + |B|^2) & -2AB \\ 2\bar{A}\bar{B} & -\varepsilon(|A|^2 + |B|^2) \end{pmatrix},$$

so

$$\begin{aligned} \Gamma &= \varepsilon g \|N\|_E = \varepsilon g ((|A|^2 + |B|^2)^2 + 4|A|^2|B|^2)^{\frac{1}{2}} \\ &= (g^2(|A|^4 + |B|^4 + 6|A|^2|B|^2))^{\frac{1}{2}}, \\ \lim_{z \rightarrow 0} \Gamma &= \sqrt{\frac{t_0^2 + s_0^2}{2}}. \end{aligned}$$

This limit is non-zero because  $a$  is non-vanishing.

Similarly, the terms  $\varepsilon g A^2$  and  $\varepsilon g B^2$  also have well defined derivatives as  $z \rightarrow \mathbb{R}$ , following from the second derivatives of  $f$ . Since  $\varepsilon g A^2$  and  $\varepsilon g B^2$  are non-zero at  $z = 0$ , their absolute values are also differentiable there. Hence the derivative  $d\Gamma$  has a well defined finite limit as  $z \rightarrow 0 \in \mathbb{R}$ .

Returning to  $\psi = 4g\Gamma$ , we have

$$d\psi(0) = \lim_{z \rightarrow 0} (4dg\Gamma + 4g d\Gamma).$$

Lemma 3.4 informs us that  $\lim_{z \rightarrow 0} g = 0$  and  $\lim_{z \rightarrow 0} \frac{\partial g}{\partial y} = \frac{4|a|(\operatorname{Im} b - r)}{H}$ , from which the claim of the lemma follow.  $\square$

**5.2. Frontals and fronts.** Let  $U$  be a domain of  $\mathbb{R}^2$ . A map  $f : U \rightarrow \mathbb{E}^3$ , into the three-dimensional Euclidean space, is called a *frontal* if there exists a unit vector field  $\mathbf{n}_E : U \rightarrow \mathbb{S}^2$ , such that  $\mathbf{n}_E$  is perpendicular to  $f_*(TU)$  in  $\mathbb{E}^3$ . The map  $L = (f, \mathbf{n}_E) : U \rightarrow \mathbb{E}^3 \times \mathbb{S}^2$  is called a Legendrian lift of  $f$ . If  $L$  is an immersion, then  $f$  is called a *front*. A point  $p \in U$  where a frontal  $f$  is not an immersion is called a *singular point* of  $f$ .

Suppose that the restriction of a frontal  $f$ , to some open dense set, is an immersion, and for some given Legendrian lift  $L$  of  $f$ , there exists a smooth function  $\psi : U \rightarrow \mathbb{R}$  such that, in local coordinates  $(x, y)$ ,

$$f_x \times f_y = \psi \mathbf{n}_E.$$



Then a singular point  $p$  is called *non-degenerate* if  $d\psi$  does not vanish there. In this situation, the frontal  $f$  is called non-degenerate if every singular point is non-degenerate.

**Lemma 5.4.** *Let  $f : \Sigma_s \rightarrow \mathbb{L}^3$  be a weakly non-degenerate generalized  $H$ -surface. Let  $\mathbf{n}_E$  denote the Euclidean unit normal defined in Section 5.1. Let  $\mathbb{E}^3$  denote the vector space  $\mathbb{L}^3$  with the standard Euclidean inner product  $\langle \cdot, \cdot \rangle_E$ . Then the map  $f : \Sigma_s \rightarrow \mathbb{E}^3$ , together with the Legendrian lift  $L = (f, \mathbf{n}_E) : \Sigma \rightarrow \mathbb{E}^3 \times \mathbb{S}^2$ , defines a frontal. The surface is non-degenerate as an  $H$ -surface, in accordance with Definition 3.5, if and only if it is non-degenerate as a frontal.*

*Proof.* By Lemma 5.1, the map  $\mathbf{n}_E : \Sigma_s \rightarrow \mathbb{S}^2$  is well defined and real analytic, and so  $L = (f, \mathbf{n}_E)$  is a real analytic Legendrian lift of  $f$ ; in particular,  $f$  is a frontal. Regarding degenerate points, the map  $\psi$  above is the signed Euclidean norm  $\epsilon \|f_x \times f_y\|_E$ , discussed in Lemma 5.3, and we showed there that  $d\psi$  vanishes at a singular point if and only if  $\text{Im} b - r$  does. The latter expression is, according to Theorem 3.3, the derivative of the function  $h$ , which was used previously to define degeneracy.  $\square$

**Lemma 5.5.** *Let  $f$  be a non-degenerate generalized  $H$ -surface constructed from the Björling data in Theorem 4.1. Then  $f$  is a front on a neighbourhood of  $z = 0$  if and only if*

$$t(0) \neq 0.$$

*Proof.* According the assumptions of the Björling construction,  $df = s(0)(-e_2 + e_3)dx + t(0)(-e_2 + e_3)dy$ . By Lemma 5.2,  $d\mathbf{n}_E = \frac{-\theta_x}{2}e_1dx + \frac{Ht(0)}{2\sqrt{2}}(e_2 - e_3)dy$ . It follows that the map  $dL = (df, d\mathbf{n}_E)$  has rank 2 at 0 if and only if  $t(0) \neq 0$ .  $\square$

**5.3. Cuspidal edges and swallowtails.** At a non-degenerate singular point, there is a well-defined direction, that is a non-zero vector  $\eta \in T_pU$ , unique up to scale, such that  $df(\eta) = 0$ , called the *null direction*.

A test for whether a singularity on a front is a swallowtail or a cuspidal edge is given in [16]:

**Proposition 5.6.** ([16]). *Let  $f : U \rightarrow \mathbb{R}^3$  be a front, and  $p$  a non-degenerate singular point. Suppose that  $\gamma : (-\delta, \delta) \rightarrow U$  is a local parameterisation of the singular curve, with parameter  $x$  and tangent vector  $\gamma'$ , and  $\gamma(0) = p$ . Then:*

- (1) *The image of  $f$  in a neighbourhood of  $p$  is diffeomorphic to a cuspidal edge if and only if  $\eta(0)$  is not proportional to  $\gamma'(0)$ .*
- (2) *The image of  $f$  in a neighbourhood of  $p$  is diffeomorphic to a swallowtail if and only if  $\eta(0)$  is proportional to  $\gamma'(0)$  and*

$$\left. \frac{d}{dx} \det(\gamma'(x), \eta(x)) \right|_{x=0} \neq 0.$$

We can use this test to prove the following result:

**Theorem 5.7.** *Let  $f$  be a non-degenerate generalized  $H$ -surface constructed from the Björling data in Theorem 4.1. Then:*

- (1)  *$f$  is locally diffeomorphic to a cuspidal edge at  $z_0 = 0$  if and only if*

$$t(0) \neq 0 \quad \text{and} \quad s(0) \neq 0.$$

- (2)  *$f$  is locally diffeomorphic to a swallowtail at  $z_0 = 0$  if and only if*

$$t(0) \neq 0, \quad s(0) = 0 \quad \text{and} \quad \frac{d}{dx}s(0) \neq 0.$$

*Proof.* By Lemma 5.5,  $f$  is a front at  $z = 0$  if and only if  $t(0) \neq 0$ , so we can use the proposition above. We also have, along  $J$ ,

$$f_x = sF_0(-e_2 + e_3)F_0^{-1}, \quad f_y = tF_0(-e_2 + e_3)F_0^{-1},$$

and the null direction is

$$(5.9) \quad \eta(x) = t(x) \frac{\partial}{\partial x} - s(x) \frac{\partial}{\partial y}.$$

Writing  $x + iy = [x, y]^T$ , the singular curve is given by  $\gamma(x) = [x, 0]^T$  and the null direction by  $\eta(x) = [t(x), -s(x)]^T$ , and so the criteria in Proposition 5.6 imply the claim.  $\square$

**5.4. Cuspidal cross caps.** From [11] (Theorem 1.4), one has the following test for whether a non-degenerate frontal is locally a cuspidal cross cap:

**Theorem 5.8.** ([11].) *Let  $f : U \rightarrow \mathbb{R}^3$  be a frontal, with Legendrian lift  $L = (f, \mathbf{n}_E)$ , and let  $z_0$  be a non-degenerate singular point. Let  $X : V \rightarrow \mathbb{R}^3$  be an arbitrary differentiable function on a neighbourhood  $V$  of  $z_0$  such that:*

- (1)  $X$  is orthogonal to  $\mathbf{n}_E$ .
- (2)  $X(z_0)$  is transverse to the subspace  $f_*(T_{z_0}(V))$ .

Let  $x$  be the parameter for the singular curve, and set

$$\tilde{\psi}(x) := \langle \mathbf{n}_E, dX(\eta) \rangle_E \big|_x.$$

The frontal  $f$  has a cuspidal cross cap singularity at  $z = z_0$  if and only:

- (A)  $\eta(z_0)$  is transverse to the singular curve;
- (B)  $\tilde{\psi}(z_0) = 0$  and  $\tilde{\psi}'(z_0) \neq 0$ .

**Theorem 5.9.** *Let  $f$  be a non-degenerate  $H$ -surface constructed from the Björling data in Theorem 4.1. Then  $f$  is locally diffeomorphic to a cuspidal cross cap around  $z = 0$  if and only if the following conditions hold:*

$$s(0) \neq 0, \quad t(0) = 0 \quad \text{and} \quad \frac{d}{dx}t(0) \neq 0.$$

*Proof.* In a neighbourhood of 0, the singular curve is given by an interval  $J = (x_1, x_2)$  of the real line. Recall from the proof of Lemma 5.2 that we found the following formula for  $\mathbf{n}_E$ :

$$\begin{aligned} \mathbf{n}_E &= \beta \text{Ad}_{e_3}(F_\omega Y F_\omega^{-1}) & Y &= ae_1 + be_2 + ce_3, \\ a &= \text{Im}(\mu)\rho Z_1, \quad b = -(\text{Re}(\mu) + \rho)\rho Z_1, \quad c = Z_1 + Z_2, \end{aligned}$$

and along  $J$  we have:  $Z_1 = \frac{1}{\sqrt{2}}, Z_2 = 0, a = 0, b = -1/\sqrt{2}$  and  $c = 1/\sqrt{2}$  so that  $Y = \frac{1}{\sqrt{2}}(-e_2 + e_3)$  for real values of  $z$ .

We will apply Theorem 5.8 with the vector field defined by the cross product:

$$\begin{aligned} X &= (\text{Ad}_{e_3}F_\omega e_2 F_\omega^{-1}) \times (\text{Ad}_{e_3}F_\omega Y F_\omega^{-1}) \\ &= -\frac{1}{2}F_\omega[e_2, ae_1 + be_2 + ce_3]F_\omega^{-1} \\ &= -F_\omega(ce_1 + ae_3)F_\omega^{-1}. \end{aligned}$$

$X$  is orthogonal to  $\mathbf{n}_E$  because  $\text{Ad}_{e_3}F_\omega Y F_\omega^{-1}$  is proportional to  $\mathbf{n}_E$ . Along  $J$  we have

$$f_x = sF_0(-e_2 + e_3)F_0^{-1}, \quad f_y = tF_0(-e_2 + e_3)F_0^{-1}, \quad X = -\frac{1}{\sqrt{2}}F_0e_1F_0^{-1},$$

so  $X$  is transverse to  $f_*(T_{z_0}(V))$ . That is,  $X$  satisfies conditions 1 and 2 of Theorem 5.8.

Now consider the conditions (A) and (B). The null direction along  $J$  is given by  $\eta = t \frac{\partial}{\partial x} - s \frac{\partial}{\partial y}$ , and this is transverse to the singular curve at  $z_0 = 0$  if and only if  $s(0) \neq 0$ , so our first condition is equivalent to condition (A).

To investigate  $\tilde{\psi}$ , we need an expression for  $\langle \mathbf{n}_E, dX \rangle_E$  along  $J$ . Now

$$dX = -c d(F_\omega e_1 F_\omega^{-1}) - a d(F_\omega e_3 F_\omega^{-1}) - d c \operatorname{Ad}_{F_0} e_1 - d a \operatorname{Ad}_{F_0} e_3.$$

Along  $J$  we have  $da = d(\operatorname{Im} \mu) \cdot \frac{1}{\sqrt{2}} \cdot 1 = 0$ , because we earlier computed  $d\mu = H t dy$  which is real. We also have  $a = 0$ , and  $\langle \mathbf{n}_E, \operatorname{Ad}_{F_0} e_1 \rangle_E = \left\langle \frac{1}{\sqrt{2}}(e_2 + e_3), e_1 \right\rangle_E = 0$ . We used that  $F_0$  takes values in  $SU(2)$  and so preserves the Euclidean inner product. Hence only the first term in the above expression for  $dX$  contributes to  $\langle \mathbf{n}_E, dX \rangle_E$ :

$$\langle \mathbf{n}_E, dX \rangle_E \big|_J = -\frac{1}{\sqrt{2}} \langle \mathbf{n}_E, d(F_\omega e_1 F_\omega^{-1}) \rangle \big|_J.$$

To compute this, we use:

$$F_\omega^{-1}(F_\omega)_x = U_\omega + V_\omega, \quad F_\omega^{-1}(F_\omega)_y = i(U_\omega - V_\omega),$$

where, from equation (3.6), at  $\lambda = 1$ ,

$$U_\omega + V_\omega = \begin{pmatrix} -2i \operatorname{Im} a + ir & 2i \operatorname{Im} a + \bar{b} \\ -2i \operatorname{Im} a + b & 2i \operatorname{Im} a - ir \end{pmatrix}, \quad i(U_\omega - V_\omega) = \begin{pmatrix} -2i \operatorname{Re} a & 2i \operatorname{Re} a - i\bar{b} \\ -2i \operatorname{Re} a + ib & 2i \operatorname{Re} a \end{pmatrix}.$$

With this and  $s = \frac{-4 \operatorname{Re} a}{H}$ ,  $t = \frac{4 \operatorname{Im} a}{H}$ ,  $b = \frac{1}{2} i H t$ , and  $r = \frac{1}{2}(\theta_x + H t)$  one obtains along  $J$

$$\begin{aligned} (F_\omega e_1 F_\omega^{-1})_x &= F_0 [U_\omega + V_\omega, e_1] F_0^{-1} \\ &= F_0 \begin{pmatrix} 4i \operatorname{Im} a - 2i \operatorname{Im} b & -4i \operatorname{Im} a + 2ir \\ 4i \operatorname{Im} a - 2ir & -4i \operatorname{Im} a + 2i \operatorname{Im} b \end{pmatrix} F_0^{-1} \\ &= F_0 \theta_x e_2 F_0^{-1}, \\ (F_\omega e_1 F_\omega^{-1})_y &= F_0 [i(U_\omega - V_\omega), e_1] F_0^{-1} \\ &= F_0 \begin{pmatrix} 4i \operatorname{Re} a - 2i \operatorname{Re} b & -4i \operatorname{Re} a \\ 4i \operatorname{Re} a & -4i \operatorname{Re} a + 2i \operatorname{Re} b \end{pmatrix} F_0^{-1} \\ &= F_0 H s (e_2 - e_3) F_0^{-1}. \end{aligned}$$

Hence we obtain the following expression along  $J$ ,

$$\begin{aligned} \tilde{\psi} &= \langle \mathbf{n}_E, dX(\eta) \rangle_E \big|_x \\ &= -\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \langle (\operatorname{Ad}_{e_3} \operatorname{Ad}_{F_0} (-e_2 + e_3), tX_x - sX_y) \rangle_E \\ &= -\frac{1}{2} \langle e_2 + e_3, t\theta_x e_2 - s^2 H (e_2 - e_3) \rangle_E \\ &= -\frac{1}{2} t \theta_x. \end{aligned}$$

Condition (B) of Theorem 5.8 is thus equivalent to the pair of equations

$$t \theta_x \big|_{x=0} = 0, \quad \left[ \frac{dt}{dx} \theta_x + t \frac{d\theta_x}{dx} \right]_{x=0} \neq 0.$$

Since  $\theta_x \neq 0$ , this pair of equations is equivalent to  $t(0) = 0$  and  $\frac{dt}{dx}(0) \neq 0$ .  $\square$

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